THE MATHEMATICAL GAZETTE

EDITED BY
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WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
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VOL. X. JULY, 1921. No. 153.

GREEK MATHEMATICS AND SCIENCE.*

BY SIR T. L. HEATH, K.C.B., K.C.V.O., F.R.S.

IT is, I think, of happy augury that it should have been found possible to arrange a joint meeting of members of the local branches of the Classical and Mathematical Associations, as well as members of the Yorkshire Natural Science Association, to hear a few words about Greek science. No doubt the classical or humane studies can stand by themselves and so (in a way) can the study of science based only on good English: but I venture to think that they are both much stronger when they stand side by side and support one another. The idea that there is any necessary antagonism or opposition of interests between them is surely unfounded, and there is room for both in our system of education. It is a mistake to suppose that in our school education there is not time for the one or the other; the question is really one of using the time that there is to the best advantage. No doubt schoolboys in the past have wasted much time in the perfunctory learning of Latin and Greek. It has often been remarked that, while a boy was more or less continuously learning Latin and Greek from the age of nine or ten till he went to the University, a girl who only began Latin and Greek at, say, fifteen or sixteen managed to catch up the boy and take a place alongside him in Part I. of the Classical Tripos. It was therefore probably the inefficient or uninteresting way in which the boy used to be taught Greek and Latin that caused the waste of time. If the subjects were made sufficiently attractive and the bogey of compulsion were removed, there is no reason why the boy of good ability should not during his school years acquire a good working knowledge of both classics and science, as well as of the indispensable amount of modern languages. And, if the boy's interest can be aroused sufficiently to make him take kindly to Greek and Latin on the one side and mathematics and science on the other, I am clear that each branch will help the other and be better done.

I will address myself first to the mathematicians. Almost all the standard terms in mathematics are Greek or Latin in origin, and, although it may be possible to explain their meaning to those who do not know Greek and Latin, the knowledge thus imparted can hardly be more than parrot-knowledge. Mathematicians will surely understand all the terms better if they know them as they arise and as part of the living language of the men who invented them. Take the word *isosceles*; a boy can be shown what an isosceles triangle is, but if he knows nothing of the derivation, he will wonder why such an outlandish term, with the attendant difficulties of spelling, should be necessary

* A paper read, March 5, 1921, to a joint meeting of the Classical Association (Leeds and District Branch) the Mathematical Association (Yorkshire Branch), and the Yorkshire Natural Science Association.

to express so simple an idea. But if the mere appearance of the word when written shows him that it means a thing *with equal legs*, being compounded of ἴσος, equal, and σκέλος, a leg, he will understand its appropriateness and will have no difficulty in remembering it. The term for an equilateral triangle, on the other hand, is borrowed from the Latin, and the boy who knows Latin will take in its meaning at once; but even this term is merely the Latin translation of the Greek ἰσόπλευρος, *equal-sided*, from ἴσος, equal, and πλευρά, a side. *Parallelogram* again can be explained to a Greekless pupil, but it will be far better understood and remembered if the boy sees in it the two words παράλληλος and γραμμή, and realises that it is a short way of expressing that the figure in question is contained by parallel lines; and again he will best understand the word *parallel* itself if he sees in it the statement of the fact that the two straight lines so described go *alongside one another*, παρ' ἀλλήλας, all the way. Similarly he will be saved from any vague idea (as a certain reviewer once put it) that a *rhombus* is a fish, or a *parallelepiped* some kind of prehistoric centipede, if he realises that a rhombus takes its name from its resemblance to a form of spinning-top (ῥόμβος from ῥέμβω, to spin), and that, just as a parallelogram is a figure formed by two pairs of parallel straight lines, so a *parallelepiped* is a solid figure bounded by three pairs of parallel planes (παράλληλος, parallel, and ἐπίπεδος, plane); incidentally, in the latter case, he will be saved from committing the monstrosity of writing "parallelopiped," a barbarism which has disfigured not a few textbooks of geometry.

Again, if we take up a textbook of geometry written in accordance with the most modern education circular or University syllabus and boasting complete emancipation from the hidebound rigidity of Euclid, we shall find that the phraseology used (except where made more colloquial and less scientific) is almost all pure Greek. The Greek tongue was extraordinarily well adapted as a vehicle for scientific thought; one of the characteristics of Euclid's language which his commentator Proclus is most fond of emphasising is its marvellous ἀκρίβεια, *exactness*. There is an apparent inevitableness about it which makes one wonder what would have happened if the Greeks had not invented science but left it for some one else to evolve. What people would have been capable of doing it, and what form would mathematics have taken? It is useless to speculate, because, in point of fact, mathematics *is* Greek through and through, and that is the end of the matter.

Having attempted to show why the mathematician should not neglect Greek if he is to read his subject intelligently, may I now appeal to the members of the Classical Association and show cause why classical students should not omit to study the Greek mathematicians? I well remember the days when, as an undergraduate, I first came across the *editio princeps* of the Greek text of Euclid (there was no handy Teubner edition in those days), and the thrill with which I read in the original Greek the familiar phrases of the propositions I had learnt at school. The reading of the Greek seemed to throw new light on Greek and new light on geometry at the same time. I am convinced that it would do every classical student good to read some portion at least of Euclid and Archimedes in the original. The Germans have already had the intelligence to see this. Von Wilamowitz-Moellendorff has included in his *Griechisches Lesebuch* extracts from the beginning of Euclid's *Elements*, from Archimedes, and from Heron of Alexandria. English editors of passages from the Greek for translation into English should follow this example.

But perhaps, after nearly forty years' study of the Greek mathematicians, I may have become a biased judge; in support, therefore, of my thesis I will quote an independent authority. And, being a Cambridge man myself, I will go for what I want to the sister University of Oxford. Members of the Classical Association will all be familiar with the brilliant, if partly contentious, book, *The Greek genius and its meaning to us*, by one of the most enthusiastic champions of Greek in the Association. The book was reviewed in the *Oxford Magazine* of 6th March, 1913. Towards the end of his review the reviewer

(I wish I knew his name) questions a remark of the author to the effect that Plato after all, "though in a thousand ways a Greek of the Greeks, in all that is most distinctive in his thought is a heretic." It may be, says the reviewer, that Plato is not direct, that he is the enemy of liberty, that he is not a humanist and that he is otherworldly. "Yet surely of all that is Greek Plato is the quintessence. To be a Greek was to seek to know; to know the primordial substance of matter, to know the meaning of number, to know the world as a rational whole. In no spirit of paradox one may say that Euclid is the most typical Greek: he would know to the bottom, and know as a rational system, the laws of the measurement of the earth. Plato, too, loved geometry and the wonders of numbers; he was essentially Greek, because he was essentially mathematical. It is true that he loved myths as well as mathematics; and it would be a long story to seek to show the affinities of the one with the other. But at any rate Plato sought to know and to understand; he demanded of every artist (be he craftsman or politician) a true knowledge of his art, and he demanded of every thinker that he should think things through and through to the Ideas which are their meaning.

"No doubt the Greek genius means many things. To one school (perhaps a prevalent school in Oxford to-day) it means an aesthetic ideal, somewhat variegated and diversified by anthropological survivals and outcrops. To others it means an austere thing, which delights in logic and mathematics; which, continually wondering and always inquisitive, is driven by its wonder into philosophy, and into inquiry about the why and wherefore, the whence and whither, of tragedy, of the State, indeed of all things. And if one thus finds 'the Greek genius' in Euclid and the Posterior Analytics, one will understand the motto written over the Academy—*μηδεὶς ἀγεωμέτρητος εἰσίτω*. To know what the Greek genius meant you must (if one may speak *ἐν αἰνίγματι*) begin with geometry; and so beginning you may arrive at what the Greeks were always seeking—some knowledge of 'the question what being is,' and, incidentally, at some knowledge of the question what the being of their genius was." I am glad to take the opportunity of commending this admirable review to the earnest attention of members of the Classical Association. I have quoted from it at this length because it expresses exactly the views which I should have wished to express but in language more vivid and eloquent than I myself could command.

We naturally ask what were the special aptitudes which the Greeks possessed for science. They had, first, a love of knowledge for its own sake amounting, as Butcher says, to an instinct and a passion; secondly, a love of truth and a determination to see things as they are; thirdly, a remarkable capacity for accurate observation. Fourthly, while eagerly assimilating information from all quarters, from Egypt and Babylon in particular, they had an unerring instinct for taking what was worth having and rejecting the rest. As one writer has said, "it remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East and threw all the fantastic rubbish on one side." Fifthly, they possessed a speculative genius unrivalled in the world's history.

It was this unique combination of gifts which qualified the Greeks to lead the world in all the intellectual pursuits that make life worth living.

Last, but not less important, the Greeks possessed the advantage over the Egyptians and Babylonians of having no priesthood which could monopolise learning as a preserve of its own, with the inevitable result of sterilising it by keeping it bound up with religious dogmas and prescribed and narrow routine. They had no Bible or Koran to interfere with the free use of their reasoning faculties. Their religion was just adequate to satisfy natural sentiment, but left free scope for the exercise of all their intellectual powers even when this tended to uproot and supplant religious dogmas.

Evidence of these things is furnished by the history of medicine in the

earliest times. Even in Homer there is hardly a trace of superstitious rites or magic formulae being applied in the treatment of wounds; the treatment is on rational lines; plants and roots are used to relieve pain, while the language employed shows so much understanding of the degree of danger attaching to a particular sort of wound, and such a surprising amount of anatomical knowledge about the most important human organs, that a German medical Chief of Staff * has actually deigned to greet the poet of the *Iliad* as a colleague. Only once (in the *Odyssey*) is a charm tried to cure haemorrhage from a wound; from this exception we may infer that the completely rational view was the prerogative of the noble and the hero, the superstitious usage in the case where it occurs betraying the bourgeois or the popular view. By the fifth century, to which period (at the latest) the Hippocratean medical writings belong, there is remarkable testimony to the complete emancipation of medicine from superstition. The term for epilepsy, which was called the *ἱερὰ νόσος*, "sacred" or "God-sent" disease, is ridiculed in uncompromising language. "Everything," says the writer, "is divine and everything is human," meaning that no one thing has a divine origin more than another. "The nature and onset of the particular disease have the same 'divine' source that everything else comes from." "These diseases appear as God-sent in precisely the same way as all others; none is more God-sent, none more human than another; ... every one of them has a nature of its own" (*i.e.* a natural cause), "and none is without such." Again, says the author, the impostors who pretend to deal with the disease put forward the "divine" origin of it as a cloak for their inability to treat it, hoping thereby to secure themselves in either event; if the patient recovers, they claim that this result is due to their cleverness and hope to acquire a reputation accordingly; if not, their defence is easy because they can protest that the fault is not theirs but that of the gods.

The Ionian Greeks who settled in Asia Minor were the most favourably situated in respect both of natural gifts and of environment for initiating philosophy and science. They had the advantage of living in contact with two ancient civilisations, the Egyptian and Babylonian, and it was what they learnt from the Egyptians and Babylonians, elementary as it was, which supplied the first impulse and set them on the road to science.

All Greek authorities are unanimous in tracing the beginnings of geometry to Egypt. It arose out of the practical needs of the country. Revenue was raised by the taxation of landed property, and its assessment depended on the accurate fixing of the boundaries of the various holdings. The landmarks were constantly being removed by the periodical flooding due to the rising of the Nile, and it was necessary therefore to replace the landmarks, or to determine the taxable area independently of them, by the aid of an art of land-surveying. Apart from the modicum of mathematics required for the construction of pyramids, geometry in Egypt was, to judge by all the surviving documents, limited to mensuration of areas and volumes from different data, and all that we find recorded is a collection of rough practical rules for measuring, first, areas such as those of a square, a triangle, a trapezium, and a circle; and secondly, the solid content of measures of corn, etc., of different shapes. The Egyptians also constructed pyramids of a certain slope by means of arithmetical calculations. The latter involved the notion of similarity of figures, especially triangles; but the whole thing was practical and went by rule of thumb.

It required the breath of a genius like that of the Greeks to give life to such soulless forms. No doubt the inspection of figures drawn to illustrate rules of measurement for circles and plane figures would suggest to Thales, who travelled in Egypt, certain similarities and congruences, and these would set him thinking whether there were not some elementary general principles underlying the construction and relations of different figures and parts of figures. This search for general principles was in accord with the Greek

* H. Frölich, *Die Militärmedizin Homers*. Stuttg. 1879.

instinct for generalisation and the attempt to find a rational explanation of everything.

Thus it was that theoretical geometry began with Thales in the first half of the sixth century B.C. The particular discoveries attributed to him are of the elementary sort which might be expected from the first founder of the science. They are, first, that a circle is bisected by any diameter; second, that the angles at the base of an isosceles triangle are equal (the famous "pons asinorum" in Euclid); third, that, if two straight lines cut one another, the vertically opposite angles are respectively equal; fourth, that, if two triangles have two angles and one side in each respectively equal, the triangles are equal in all respects. Fifthly, Thales is said to have been the first to inscribe a right-angled triangle in a circle: which must mean that he was the first to discover that the angle in a semicircle is a right angle. He also solved two problems of a practical kind: first, he showed how to measure the distance of a ship at sea; secondly, he measured the heights of pyramids by means of the shadows thrown on the ground by the pyramid and by a stick at the same moment; choosing the time when the length of the stick itself and of its shadow were equal, he argued by similarity of triangles that, at the same instant, the length of the shadow of the pyramid gave its height also.

Greek astronomy also begins with Thales. Everyone knows the story of his falling into a well when star-gazing and being rallied by a clever and pretty maidservant from Thrace (as Plato has it) for being so eager to know what goes on in the heavens that he could not see what was straight in front of him, nay, at his very feet. But he was not merely a star-gazer. There is good evidence that he predicted a solar eclipse which took place on 28th May, 585 B.C. We can conjecture the basis of this prediction. The Babylonians, as the result of observations continued through centuries, had discovered the period of 223 lunations after which eclipses recur; and Thales had probably heard of this period, either directly, or through the Egyptians as intermediaries. Thales seems further to have discovered the inequality of the four astronomical seasons; works on the Solstice and on the Equinox are attributed to him. He used the Little Bear in preference to the Great Bear as a means of finding the pole.

No one writing or speaking on Greek science should omit to notice the bold flights of the genius of Thales's immediate successor in the list of Ionian philosophers, I mean Anaximander. Anaximander wrote one treatise, *On Nature*, in which doubtless he tried to explain everything in heaven and earth. Among his strikingly original ideas the following should be mentioned. The first principle of all things he took to be, not water, or air, or fire, or any of the elements, but another sort of substance which he called by the name of the Infinite (ἄπειρον) without further definition. From this he held that the heavens are separated off and, generally, all the worlds, which are infinite in number. There are, he said, always some worlds coming into being and others passing away; they pass away into that from which they arise, "for they must pay the penalty and make reparation for the injustice they have committed, according to the sequence of time."

Anaximander boldly maintained that the earth is in the centre of the universe, suspended freely and without support, and that it remains where it is because it is equidistant from all the rest of the heavenly bodies, by which he no doubt meant that the whole universe formed a system in equilibrium, as Plato afterwards held. The earth is like a tambourine, or a short cylinder, with two circular plane faces, on one of which we live; the length of the cylinder is one third of its breadth.

The heavenly bodies are circular hoops of compressed air, opaque but containing fire inside, which shines out through vents, like gas-jets; their motion is the rotation of the hoops about their centres. It has been pointed out that this idea of the formation of tubes of compressed air in which the fire of each star is shut up except for the one opening is not unlike Laplace's hypothesis about the origin of Saturn's rings.

As if this were not enough, we have a suggestion about the origin of life which reminds us of Darwin. According to Anaximander animals first arose from slime evaporated by the sun; they at first lived in the sea and had prickly coverings; men at first resembled fishes.

Lastly, Anaximander was the first to draw a map of the inhabited earth, which, with improvements made later by Hecataeus, became an object of general admiration.

To return to the mathematical subjects. Next in succession to Thales came Pythagoras and the Pythagoreans. The name of Pythagoras was one to conjure with, and it came to be the practice to refer all Pythagorean discoveries, more or less, to the founder of the school (*αὐτὸς ἔφα, ipse dixit*). Hence it is difficult to be sure how much of the Pythagorean mathematics was due to Pythagoras himself. But there are certain things that may safely be attributed to him. To begin with astronomy. Pythagoras was the first to hold that the earth is spherical in shape, and he probably took the same view with regard to the other heavenly bodies; a powerful motive to this was no doubt the conviction that the sphere was the most beautiful of all solid figures. Pythagoras is further said to have recognised the identity of the Morning and Evening Stars, and the fact that the sun, moon and planets have an independent movement of their own in a sense opposite to that of the daily rotation.

Arithmetic in the sense of the theory of numbers began with Pythagoras. This theory was geometrical in form, numbers being commonly represented by dots forming and filling up geometrical figures such as lines, squares, rectangles, polygons, and again solid figures such as pyramids, cubes and parallelepipeds. There is a possible explanation of this point of view. The Pythagoreans were keen astronomers, and it would probably be borne in upon them, as they continually watched the heavens, that in each constellation there are a certain number of visible stars forming a certain figure. This may account for the expression "the number in the heaven" used by the Pythagoreans, and perhaps even for their view that the whole heaven is numbers; and again the identification of the stars with points in figures may explain the statement of Aristotle that the Pythagoreans regarded points as having magnitude.

The most wonderful discovery about numbers made by Pythagoras is that of the dependence of the musical intervals upon numerical ratios. Pythagoras discovered that, with strings at the same tension, a difference of length in the ratio of 2 to 1 gives the octave, the ratio of 3 to 2 the fifth and that of 4 to 3 the fourth. It is not surprising that, in view of connexions of this sort between numbers and natural phenomena, the Pythagoreans came to liken all things to numbers and to find in the principles of numbers the principles of all things.

Coming to geometry, we find Proclus saying that Pythagoras transformed the study of geometry into a liberal education, examining the principles of the science from the beginning. That is to say that he was the first to systematise geometry as a subject in and for itself, beginning with definitions and the other necessary preliminary assumptions. Pythagoras, we are told, called geometry by the general name "inquiry," as if implying that geometry was the whole of philosophy. There is a characteristic story that, when he returned to Samos from his travels in Egypt and tried to found a school, the Samians were indifferent, and he had therefore to take special measures to ensure that geometry should not perish at its birth. Going to the gymnasium, he sought out a well-favoured youth who seemed likely to suit his purpose and was withal poor, and bribed him to learn geometry by promising him sixpence for every proposition that he mastered. Very soon the youth became fascinated by the subject, and Pythagoras rightly judged that the sixpenny inducement was no longer necessary. He hinted therefore that he himself was poor and must try to earn his living instead of doing mathematics; whereupon the youth, rather than give up the study, offered to pay Pythagoras sixpence for each proposition.

Of the discoveries commonly attributed to Pythagoras the most famous is of course the theorem of Euclid I. 47, that, in any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. Every one knows the story that Pythagoras sacrificed an ox in celebration of this discovery. The authorities are indeed somewhat late, and all the different versions seem to be ultimately traceable to a distich by one Apollodorus (date unknown):

> When Pythagoras discovered that famous proposition on the strength of which he made a splendid sacrifice of oxen.

Plutarch, it is true, expresses doubt whether the occasion of the sacrifice was the discovery of this proposition or of another, namely, the solution of the problem, given two rectilineal figures, to draw a third rectilineal figure equal in area to one of the given figures and similar to the other. But the remarks of Plutarch do not suggest any doubt that Pythagoras was the discoverer of both propositions.

The Egyptians, long before Pythagoras's time, were aware that a triangle with sides in the ratio of the numbers 3, 4, 5 is right-angled, and it is believed that this and other right-angled triangles with sides in rational numbers were known in India as early as the eighth century B.C. But there is no sufficient evidence that even the Indians invented any general proof; and, even if they did, it is probable that Pythagoras's discovery was independent. The truth of the tradition is to some extent confirmed by another which attributes to Pythagoras a general formula for finding any number of right-angled triangles with sides in the ratio of rational numbers. This was no doubt evolved through the theory of the *gnomon*, a term which must have seemed mysterious to many a generation of schoolboys reading Book II. of Euclid's *Elements*. Originally meaning the upright needle of a sundial, enabling anyone to recognise (γιγνώσκω) or read the time of day, the word *gnomon* next came to be used of a figure in the shape of a carpenter's square, the figure which, when put round two sides of a square, makes up a larger square. The gnomon had a corresponding arithmetical signification; it was part of the geometrical arithmetic of the Pythagoreans. Imagine the number 1, the unit, represented by a dot; place three dots round it so as to make the four dots the corners of a square; the three dots so added form the first gnomon. Next place five dots round two sides of the square formed by the four dots which has two for its side, and we have a square with three in its side. $1+3=4$; $1+3+5=9$; 5 is the second gnomon; and so on. In fact all the successive odd numbers, 3, 5, 7, etc., are gnomons, and the sum of any series of successive odd numbers beginning with 1 is a square number. Any odd number is thus the difference between some square number and the next higher integral square, that is, the square number having one more in its side: 7 is the difference between the squares 9 and 16, 11 the difference between the squares 25 and 36, and so on. It is easy by means of this fact to deduce Pythagoras's formula for finding right-angled triangles with sides in rational numbers.

Pythagoras is credited with two other discoveries, the theory of proportionals and the construction of the so-called "cosmic figures," the five regular solids. The theory of proportionals in question was a numerical theory such as is expounded in Euclid's Book VII. Connected with proportion is the theory of means, three of which, the arithmetic, geometric and sub-contrary (afterwards called harmonic) were known to Pythagoras.

Among the so-called "cosmic figures" is the dodecahedron, the solid with twelve equal regular pentagons as faces. The construction of this solid involves that of the regular pentagon itself, and this again depends on the solution of the problem of cutting a straight line in extreme and mean ratio (as it is called), *i.e.* so that the whole straight line has to the greater of the two segments the same ratio as the greater segment has to the lesser. This problem is solved in Prop. 11 of Euclid, Book II., and is a simple case of a

general method invented by the Pythagoreans and known as the *application of areas*. The simplest case of all is that of Euclid I. 44, 45, where it is required to apply to a given straight line as base a parallelogram which shall have a given angle and shall be equal in area to any given rectilineal figure. In the more difficult cases the parallelogram to be applied has to overlap or to fall short of the given straight line, but so that the portion overlapping or the portion which is deficient has a given shape. Application pure and simple is παραβολή, application with an overlap is ὑπερβολή and application with a deficiency is ἔλλειψις. The three varieties of the process of application are used by Apollonius of Perga in finding the fundamental property of the three conic sections respectively, and this is the origin of the three names *parabola*, *hyperbola* and *ellipse* which Apollonius was the first to give to the three curves.

The Pythagoreans, then, originated two out of the three great methods running through the whole of Greek geometry, namely, those of proportion and the application of areas. Their theory of proportion, it is true, being numerical, was inadequate in that it did not apply to incommensurable magnitudes; and hence geometrical theorems proved by it were not proved universally. But, subject to this qualification, we may say that the Pythagorean geometry covered the bulk of the subject-matter of Books I., II., IV. and VI. of Euclid's *Elements*. The case is not so clear with regard to Book III. of the *Elements*; but, as the main propositions of that Book were known to Hippocrates of Chios in the second half of the fifth century B.C., we may be sure that they too were part of the Pythagorean geometry.

But the Pythagoreans also discovered the existence of incommensurable lines, or of irrational. This was no doubt first discovered with reference to the diagonal of a square, which is incommensurable with its side, being in the ratio to it of $\sqrt{2}$ to 1. The Pythagorean proof of this particular case is quoted by Aristotle; it is by a *reductio ad absurdum* proving that, if the diagonal is commensurable with the side, it will follow that one and the same number is both odd and even. At the same time, by means of a proposition afterwards incorporated in Euclid's Book II., the Pythagoreans showed that, although you can never express the ratio of the diagonal of a square to its side as the ratio of one number to another, you can obtain a series of arithmetical ratios approximating more and more closely to the ratio in question, and can carry the approximation as far as you please.

No less striking were the speculations of the Pythagoreans in astronomy. Pythagoras himself, while attributing spherical shape to the earth and the other heavenly bodies, and recognising the independent motions of the planets, seems to have retained the earth in the centre. His successors actually abandoned the geocentric theory and made the earth one of the planets moving like them round a common centre. The new system only fell short of the Copernican in that the planets did not revolve round the sun as centre, but the sun itself, like the moon, the earth and the planets, revolved about the central fire in which was supposed to reside the governing principle and the force which directs the movements and activity of the universe. The Pythagoreans also assumed another body revolving round the central fire between it and the earth, but always in company with the earth, called the counter-earth; this, like the central fire, was invisible to us because the hemisphere of the earth on which we live was supposed always to face outwards, away from the central fire. The order of the revolutions starting from the central fire was counter-earth, earth, moon, sun, then the five planets and, outermost of all, the sphere of the fixed stars. This made ten revolutions in all. Ten being the perfect number according to the Pythagoreans, some supposed that the counter-earth was invented solely in order to make up that number, but it is more probable that the object was to explain the frequency with which eclipses of the moon occur as compared with those of the sun.

Nor did the Pythagoreans confine themselves to mathematics and astronomy; they occupy an important place in the history of medicine. There was a

flourishing school of medicine at Croton, and important discoveries were made by Alcmaeon of that place. Alcmaeon dissected animals, and discovered the most important of the sensory nerves (which, like Aristotle, he regarded as tubes or ducts) and their connexion with the brain, in which he placed the centre of perceptive and intellectual life. He explained illness and disease as the disturbance of the equilibrium between the elementary opposites in the body, the cold and warm, the dry and moist, etc. In this Alcmaeon was a forerunner of Hippocrates. Alcmaeon made the separate senses the subject of close researches; he tried to explain the fact that when we are hit in the eye we "see sparks" by the hypothesis that the eye contains fire in it. He relied for proofs upon definite evidence: about things invisible as well as about things mortal only the gods (he said) can have certain knowledge; we, being human, must rely on inference from evidence (τεκμαίρεσθαι).

Two other natural philosophers should before now have been mentioned for definite contributions to science, Anaxagoras (500-428 B.C.) and Empedocles (494-434 B.C.). Anaxagoras discovered that the moon receives its light from the sun; the moon according to him is made of earth, the sun is a red-hot stone larger than the Peloponnese. He had original ideas on the evolution of the universe. Assuming a rapidly rotating mass in the centre, he conceived that the violence of the whirling motion caused portions to be detached and hurled off into space; these hurlings-off are very like those by which Kant and Laplace accounted for the formation of the solar system. Anaxagoras also gave a tolerably accurate explanation of the risings of the Nile. Empedocles deserves mention for his theory that light travels, taking time to pass from one point to another. Aristotle mentions the theory, but dismisses it as too bold an assumption.

Nor should I omit to mention the great name of Democritus. Of his doctrine of atoms a Frenchman has remarked that "None of the ideas which antiquity has bequeathed to us has enjoyed a greater or even an equal triumph." Democritus may indeed be called the father of Physics. In mathematics he was no less distinguished, as we shall see later on.

Of other subjects I am not qualified to speak in detail. But on the biological side I ought to refer to Aristotle who was the creator of scientific zoology and comparative anatomy.* Further, his pupil Theophrastus was the first to write systematic treatises on botany. Two elaborate works of his on plant lore and plant physiology respectively enable us to claim that the Greeks founded the science of botany also.

In speaking of mathematics I have confined myself to the Elements. I should have liked to say something of the higher geometry of the Greeks: how, in the century from 450 to 350 B.C., when the Elements were not yet fully established, they had advanced beyond the Elements and solved three famous problems in higher geometry, the squaring of the circle, the trisection of any angle, and the duplication of the cube; how Eudoxus, a pupil of Plato, discovered, first, the great theory of proportion incorporated by Euclid in his Book V. and, secondly, the "method of exhaustion" (as it is called) by means of which Eudoxus was able to measure the content of pyramids and cones and to deal with the areas of circles and the volumes of spheres; how Archimedes, using and extending Eudoxus's method, measured curvilinear areas and solids by a procedure which, though purely geometrical, is equivalent to the integral calculus; how, again, in astronomy Heraclides of Pontus, a pupil of Plato, discovered that the earth rotates on its own axis and that Venus and Mercury revolve round the sun like satellites, and how finally Aristarchus of Samos, about 25 years earlier than Archimedes, put forward the heliocentric hypothesis, thereby completely anticipating Copernicus.

* Cp. L. C. Miall, *Early Naturalists*, p. 3, "Aristotle is the real founder of Comparative Anatomy, and perhaps no science ever made so prosperous a start, enriched from its birth with such a multitude not only of facts but of ideas."

All this, however would make a long story and would perhaps be unsuitable to a mixed audience.

What I should like now to emphasise is the character of the Greek contribution to science which makes it a possession for all time. To put it shortly, the Greeks created science, and mathematics in particular, by laying down principles, fixing terminology and inventing the methods *ab initio*; and they did this with such certainty that in the centuries which have followed there has been no need to go back on anything essential or to unlearn anything that they taught.

On the principles no one could be clearer than Aristotle. Every demonstrative science, he says, has to do with three things, the subject-matter, the things proved and the things from which the proof starts (ἐξ ὧν). It is not everything that can be proved, otherwise the chain of proof would be endless; you must begin somewhere, and you must start with things admitted but indemonstrable. These are, in the first place, principles common to all sciences which are called *axioms* or *common opinions*, as that "of two contradictories one must be true," or "if equals be subtracted from equals the remainders are equal"; secondly, principles peculiar to the subject-matter of the particular science, say, geometry. First among the latter principles are definitions: there must be agreement as to what we mean by certain terms. Secondly, we must assume the existence of certain things, *e.g.* in geometry of points and lines, in arithmetic of the odd and even. Thirdly we must assume certain other things which are less obvious and cannot be proved but yet have to be accepted, at least in the first instance, on the *ipse dixit* of the teacher only; these are called *postulates*, because they make a demand on the faith of the learner. Euclid's Postulates are of this kind, especially the fifth, known as the parallel-postulate.

Definitions began with Pythagoras. Plato gave much attention to them. It is clear that certain axioms and postulates were formulated before Aristotle's time. When therefore we come to the definitions, postulates and axioms or "common notions" in Euclid, we have a collection which is largely traditional, and only in a few cases, as in that of the definition of a straight line, and that of the parallel-postulate, can we be fairly certain that they were due to Euclid himself. It is however in Euclid that terminology is fixed for the first time; in Plato and Aristotle it was still somewhat fluid. The first terms for things geometrical were based on physical resemblances to common objects; *e.g.* an *angle* was γλωχίς, the word for an arrow-head, *surface* was χροιά, colour or skin, *perpendicular* was a straight line drawn *gnomon-wise* (κατὰ γνώμονα), *i.e.* like the perpendicular needle of a sundial, *point* was στιγμή, a *puncture*. The terms became more abstract in course of time: σημεῖον, a mark, replaced στιγμή as the name for a point, *surface* became ἐπιφάνεια, signifying what is apparent to the eye. Certain terms which had at first a more general signification became restricted in use; *e.g.* τετράγωνον is still used in Aristotle for any figure having four angles (as the name implies), *i.e.* a quadrilateral; with Euclid it definitely becomes a *square*. And so on.

We come now to methods and form of exposition. Take a proposition of Euclid and compare it with a proposition in a modern mathematical textbook; if there is any difference, it is that, if anything, the old Alexandrian was more completely rigorous than the writer of to-day. In a proposition of Euclid there were definite subdivisions, afterwards called by specific names, which were arranged in strictly logical order. There was first the *enunciation* (πρότασις); secondly, the *setting-out* (ἔκθεσις), *i.e.* the statement of the precise data, a given straight line, two triangles or whatever they might be; thirdly, the so-called διορισμός, literally *definition* or *delimitation*, which is a statement of what we are required to do or to prove, not in general terms as in the enunciation, but with reference to the particular data in the setting-out; then, fourthly, comes the *construction* (κατασκευή), the addition, where necessary, of more lines, etc., to the figure; fifthly, the *proof* (ἀπόδειξις), and sixthly,

the *conclusion* (συμπέρασμα), stating what has actually been done, which in general follows the wording of the original enunciation. In the case of a problem it often happens that a solution is not possible unless the particular data are such as to satisfy certain conditions; in this case there is yet another constituent part in the proposition, namely a statement of the condition or conditions of possibility, which was called by the same name διορισμός, delimitation, as that applied to the third constituent part of a theorem.

Now this elaboration in the arrangement of propositions was not the work of Euclid alone. It was the result of practical evolution of logical method. The earliest complete textbook that we possess is by Autolycus, one of Euclid's immediate predecessors. The propositions in it are precisely in the Euclidean form. Earlier still are certain proofs preserved by later writers; one is by Archytas (nearly a century before Euclid), others are contained in the fragments from Hippocrates of Chios (second half of fifth century B.C.) preserved by Eudemus, some more are Pythagorean; in all these cases the form of proof is essentially the same, showing that it goes back to the Pythagoreans, if not to Pythagoras himself.

The technical terms for particular methods were probably invented later, after the methods themselves had become established.

One method of solution was the *reduction* of one problem to another. This was called ἀπαγωγή, a term which seems to occur first in Aristotle. But instances of such reduction occurred long before. Hippocrates of Chios reduced the problem of duplicating the cube to that of finding two mean proportionals in continued proportion between two straight lines, that is, he showed that, if the latter problem could be solved, the former was thereby solved also; and it is probable that there were still earlier cases in the Pythagorean geometry. Next, there is the method of mathematical *analysis*. The invention of this is specifically attributed by Proclus and Diogenes Laertius to Plato; but this is more than doubtful, because like *reduction* (to which it is closely akin) analysis in the mathematical sense must have been in use much earlier. *Analysis* and its correlative *synthesis* are carefully defined by Pappus: "in analysis we assume that which is sought as if it were already done and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of principles. But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis, and by arranging in their natural order as consequences what were before antecedents and successively connecting them one with another we arrive finally at the construction of that which was sought."

Again, the method of *reductio ad absurdum* is a variety of analysis. We start from a hypothesis, namely the contradictory of that which we desire to prove: then we use the same process of analysis, carrying it back until we arrive at something admittedly false or absurd. Aristotle describes this method in various ways as *reductio ad absurdum*, proof *per impossibile*, or proof leading to the impossible. But here again, though the term was new, the method was not. Aristotle in fact illustrates it by reference to the ancient (probably Pythagorean) proof that the diagonal of a square is incommensurable with its side: the proof shows (as we said) that, if the diagonal were commensurable with the side, it would follow that one and the same number is both odd and even, which is absurd.

The story of Greek mathematics to the time of Archimedes and Apollonius is a record of an almost continuous series of capital discoveries following each other in rapid succession, and of successful attacks upon one problem after another, each more difficult than the one before, culminating within about three centuries from the first beginnings. But it must not be supposed that it was all plain sailing: far from it. The genius of the Greeks for clear thinking and carrying everything through to its logical conclusion showed itself just as much in criticism as in construction; in criticism too they seem to have

"thought of everything," as Aristotle says of Democritus. Greek mathematics in fact passed through the fire. So accustomed were mathematicians to having to defend themselves against criticism that they had a technical term for the objections which were or might be raised to their demonstrations; *ἔνστασις* was the word. There is at least one proposition, or rather a part of a proposition, in Euclid which, according to Proclus, was inserted for no other purpose than to enable an objection to be met which might be raised against a later proposition on the ground that it appeared to take account of one case only where two cases were possible.

The first great crisis through which Greek geometry passed was the result of the discovery of the incommensurable by the Pythagoreans. This invalidated the proofs of all theorems about magnitudes incommensurable with one another which used the old theory of proportion invented by the Pythagoreans, since that theory, being a numerical theory, applied only to commensurable magnitudes. That method of proof had therefore to be abandoned in such cases in favour of some other not depending on proportions. It is probable that many proofs were so remodelled. Although, when this could be done, the propositions were rehabilitated, great inconvenience was caused and a slur cast on geometry generally. No wonder that the discovery of the incommensurable was supposed to have been kept secret for a time and that death by shipwreck was said to have punished the first person who divulged it. The scandal was not finally removed till Eudoxus established his great theory of proportion which was applicable to incommensurable and commensurable magnitudes alike.

Closely connected with the difficulty of the incommensurable was that of continuity and the infinite divisibility of continuous magnitudes. Democritus stated the dilemma in a striking form in his question about successive sections of a cone parallel to the base. Suppose, he said, you have a section of a cone parallel to and (as we should say) indefinitely close to the base—the very next section to the base, so to speak—what are we to say of the section? Is it equal to the base or not equal to it? If it is equal, the next section will likewise be equal to the same and the cone will really be, not a cone at all, but a cylinder. On the other hand, if the next section to the base is unequal to the base and in fact less, and if the next succeeding section is again less, the cone will have indentations on its surface like steps, which is very absurd.

Even more piquant were the paradoxes of Zeno which have just as much point to-day as they had twenty-three centuries ago, and which (if we are to believe Mr. Bertrand Russell) were never even properly understood till recent years. Aristotle is said to have called Zeno the inventor of dialectic, and no wonder. His four puzzles took the form of proving by *reductio ad absurdum* that motion is impossible. They are complementary to one another; two are based on the hypothesis that time and space are divisible *ad infinitum*, two on the hypothesis that both time and space are made up of indivisible elements, and both hypotheses are shown to lead to the same result, that motion is impossible. In other similar arguments which are handed down by Simplicius, Zeno held that, if divisibility is without limit, it must end, not in the infinitely small (because that is by hypothesis still divisible), but in *nothing*. But, if you reverse the process and add nothing to itself for ever and ever, you will never produce any finite magnitude at all. This proof that the infinitesimal or infinitely small does not exist had a profound influence on the later developments of mathematics. Antiphon the Sophist, for instance, had said that, if you inscribed successive regular polygons in a circle, continually doubling the number of sides, you would sometime arrive at a polygon the sides of which would coincide with the circumference of the circle. Warned by Zeno, mathematicians denied this and substituted the statement that, by continuing the construction you could inscribe a polygon approaching equality with the circle as nearly *as you please*. Similarly in all cases they avoided the expressions infinitely great and infinitely small, confining themselves to saying

that a magnitude could be increased until it exceeded any given finite magnitude of the same kind however great, and conversely that a magnitude could be diminished by subtraction of half or more than half, then of half or more than half the remainder, until there was left a magnitude smaller than any assignable magnitude of the same kind however small.

In the time of the Sophists mathematics had its ups and downs. Some of them, like Hippias of Elis, were distinguished in mathematics and taught it regularly. Others would have none of it. Plato represents Protagoras as saying "The other Sophists maltreat the young, for at an age when the young have escaped from the arts, they take them against their will and plunge them once more into the arts, teaching them the arts of calculation, astronomy, geometry and music"—and here he cast a glance at Hippias—"whereas, if any one comes to me, he will not be obliged to learn anything except what he comes for." Protagoras too attacked the mathematicians on their own ground; for he wrote a book on mathematics, περὶ τῶν μαθημάτων, and, argued, against the geometers, that no such straight lines or circles as they assume exist in nature and that, *e.g.*, a material circle does not in fact touch a straight line at one point only. We hear of a title of a book by Democritus "On a difference of opinion (if that is the right reading) or on the contact of a circle and a sphere"; and it is tempting to suppose that the object of the book was to refute attacks on geometry such as that of Protagoras. Aristotle too has much the same kind of criticism in mind when he says "It is untrue to assert, as some do, that a geometer's hypotheses are false because he assumes that a line which he has drawn is a foot long when it is not, or straight when it is not straight. The geometer bases no conclusion on the particular line being what he has assumed it to be; he argues about what it represents, the figure being merely an illustration."

The Epicureans and Sceptics objected to the whole of mathematics; the basis of their objection was almost entirely the opposition between the fundamental hypotheses of mathematics, the point, the line, the surface, etc., and the data of sense-perception. There are no such things (they said) as mathematical points, lines, etc. Even if points exist, you cannot make up a line out of points. It is absurd to define a line as that which, if it is turned about one of its extremities, will always touch a plane; a line being, say, length without breadth, and therefore an unsubstantial thing, cannot be turned round at all. And so on.

Lastly, there were the critics who, without objecting to the subject-matter or the methods of mathematics, objected that the assumptions made did not cover the whole ground, in other words, that the proofs contained *tacit* assumptions, which were not admissible without some sort of argument. The very first proposition of the first Book of Euclid was assailed on this ground by Zeno of Sidon. Euclid draws an equilateral triangle by joining the two ends respectively of the given straight line to the point in which the two circles cut one another. The proof that this gives an equilateral triangle is not conclusive (said Zeno) unless it be first assumed that neither two straight lines nor two circles can have a portion common to both. Suppose for instance that the two straight lines meet before they reach the point of intersection of the circles; then we have, not an equilateral triangle, but an isosceles triangle with the two equal sides less than the base and a bit of a straight line standing up at the top like a flagstaff. Even Zeno's criticism was taken so seriously that Posidonius wrote a whole book to controvert him. Thus it was that Greek mathematics, having had its procedure checked and controlled at every point, became the beautiful and perfect thing that we find it.

There is a story that King Ptolemy asked Euclid for a short cut to geometry, and that Euclid replied that "in geometry there is one road for all." It is surely true to say that there is only one road to science and that it was the Greeks who found it.

SELECTION IN ARITHMETICAL EXAMPLES.

SELECTION plays an important part in the life of the individual. It is a fundamental process which he must be able to apply in order to understand the world into which he is born. An individual placed amidst any set of circumstances does not assimilate the whole of them. He does not act like a camera plate, which reproduces automatically every feature of an external situation. The complex mental image which he forms from his surrounding circumstances is not an exact copy of the original environment.

In any situation some circumstances, on account of their nature, may stand out more prominently than others. On a gala day in a town, the decorations, the procession and the noise of the band are distinctive features. They are inherently more noticeable than other features in the town, and they, in particular, would impress themselves on a more or less passive spectator. But there is another factor in determining what constituents of a person's surroundings shall influence him. He himself may choose which elements he will assimilate. He may disregard procession, band and decorations on a gala day and wander round simply noting one thing here and another there in a discursive sort of way. Or he may make a more purposive selection; he may be concerned to describe the affair from a particular point of view, *e.g.* as a relic of the mediaeval fair or as a mode of popular enjoyment, in which case he will select and group together the elements which serve to illustrate his purpose. The portion of his surroundings which is to survive as part of his experience is largely determined by his mental outlook.

Any situation then consists of elements which may be associated in groups, each of which gives a meaning to the situation. To grasp a situation thoroughly is to see the elements in all their inter-relations and to apprehend the meanings implied in all their associations; this involves selection in the first place. Selection, therefore, is a fundamental process.

In the early days of civilisation, when facts were few, knowledge meagre and the activities of life simple, any situation in which an individual became involved was necessarily uncomplicated. The child therefore could be gradually trained to assimilate his environment merely by sharing in the life of his elders. But as higher stages of development were reached and civilisation increased in complexity, this became impossible. The gap between the child's capacity and the achievements of the race became too great. Hence schools arose wherein the products of civilisation were presented to the child piecemeal. Such is the condition of affairs to-day. The facts and circumstances of the world at large are too complicated and too abstruse for the immature mind of the child. Things are too vague and undifferentiated. Situations are too full of incident. He lacks the knowledge and power to select related elements from his environment and apprehend their collective implication. He cannot see wood for trees. The school exists to meet this difficulty. In school only such portions of racial knowledge and experience are placed before him as are suitable to his stage of development. Selection is done for him. Eventually, however, he must select for himself, and one function of the school is to train him in the process of selection. Practice in selection therefore should be an important feature of school work.

Mathematics, in its early stages, is one of the best subjects for training the immature mind in selection, since the facts dealt with are few and uncomplicated. Thus, in forming a number concept, he must fix his attention on one element of a situation only. He sees seven boys, seven desks, seven strokes, etc., and must select the common element, viz. the "seven-ness" in each set of circumstances. It is because the number of irrelevant elements in sticks or strokes is a minimum that these articles are particularly suitable for number teaching, *i.e.* for the cultivation, in its early stages, of a quantitative outlook on circumstances.

In higher work selection of this nature is not frequent. Probably it is most commonly met with in Geometry, where it is often necessary to isolate mentally some lines and angles from the rest of the figure. But in this case, of course, the situation from which selection is made is not a natural one, *i.e.* does not involve circumstances taken from everyday life.

Arithmetic offers a most useful field for training in selection, and could well be developed on lines furthering this aim more than it is furthered at present. Arithmetical teaching is still suffering from the old conception which required a certain proficiency in carrying out mechanical processes and applying them to standardised problems. The cultivation of a mathematical outlook on the facts and circumstances of everyday life as an aim of arithmetical teaching has not yet come into its own. Quantitative considerations now enter largely into most of the circumstances of life. Ability to deal with statistics of various kinds, in particular, is becoming increasingly desirable. The pressing problems of the day are being considered more from a mathematical point of view. If the individual is to be able to view matters in this light, he must be trained to do so. To appreciate a situation in its mathematical aspect he must select the elements which have a mathematical bearing on each other and apprehend them in this relation. This means that the old type of arithmetical example, which provided for training only in appreciating the mathematical relation between facts already selected, is insufficient. The arithmetical problem should involve selection of data as well.

It has been urged against problems of this nature that they involve the very difficulty which it is the purpose of the school to remove, *i.e.* that they are too complicated and tend to produce confusion in the mind of the pupil. If, however, the concrete material from which they are formed is carefully watered down to suit the capacities of the pupils, this danger disappears. Some years ago experiments were tried in setting examples involving selection at a series of examinations for Entrance Scholarships to Secondary Schools. A few of the examples are given below, and comments taken from or based on the examiner's reports are added.

Candidates aged 12 *and* 13.

I. A man pays £3. 16*s.* 6*d.* in rates. How much of this is spent on education if each 4*s.* 6*d.* is spent as follows:

		s.	*d.*
Poor Rate		1	$0\frac{1}{2}$
Highway Rate			$10\frac{1}{2}$
City Rate—	*d.*		
Police Department	4·05		
Education Department	15·97		
Other Departments	10·98		
	31·00	2	7
Total		4	6

Comment.—In this case only a few candidates failed to select the Education Rate, 15·97*d.*, correctly.

II. If the figures in the table below are correct, which of the statements is inaccurate? How would you alter it so as to make it accurate?

Kind of food.	Percentage of water.	
Fish	75	When you buy *fish* ninepence out of every shilling goes for water.
Bread	39	When you buy *bread* fourpence out of every shilling goes for water.

Comment.—This question offered little difficulty to the majority of the candidates. Very few selected the *four* given numbers, *e.g.* 100, 75, 1*s.*, 9*d.*, and showed that they were proportional. The general method was to calculate one of the four from the remaining three.

III. A copy of a butcher's bill is given below. Find the price per pound of the neck of mutton.

	s.	*d.*
Shin of beef, 2 lbs. 2 ozs. Neck of mutton, 3 lbs. - - - - - - -	4	11½
Sirloin, 5½ lbs. Tripe, 1 lb. - - - -	7	0
Sausage, 1 lb. - - - - - -		9
Neck of mutton, 2 lbs. 8 ozs. Tripe, 1 lb. Cow-heel, 6*d.* - - - - - -	3	7
Sirloin, 5½ lbs. - - - - - -	6	1½
	£1	2 5

Comment.—Many failed to select the last line as the starting-point of the solution. The commonest incorrect method was to select the wrong line and ignore the difference in the kind of meat, *e.g.* item 1 was taken as 5 lbs. 2 ozs. for 4*s.* 11½*d.* Another method was to assume a price for one item, such as the tripe in line 2, and hence obtain a price for the other item.

IV. A copy of an advertisement is given below. Find out whether the advertiser's calculation is accurate, and if not correct it.

> 1,190 per cent. Profit from Poultry.
> How 24*s.* became £15. 10*s.* 0*d.*

Comment.—This was done moderately well, three of the four given numbers being selected in practically all cases. Many performed the required calculation but failed to correct the statement.

Candidates aged 10 *and* 11.

I. A man uses a penny-in-the-slot gas meter and burns the same amount of gas each day. How many pennies per day must be put in the meter to obtain the gas charged for in the bill below ?

Mr. J, Smith

Dr. to the Barton Gaslight Company,
For Gas during the Quarter ending June 30th, 1911.

		£	*s.*	*d.*
State of Meter -	14300 *c. ft.*			
Last Index - -	5200 ,,			
Quantity consumed -	9100 ,, at 2/6 *per* 1000 *c. ft.* - -	1	2	9

Comment.—A large number of candidates obtained full marks. Solutions involving cubic feet were very frequent. Many candidates ignored the given date, June 30th, and obtained the number of days in a quarter by dividing 365 by 4. Many made a more or less neat copy of the bill as set out in the question; usually they failed to score marks.

II. Below is given a drawing, to scale, of a clock-face without minute hand. Find, as nearly as you can, what time is shown by the clock. Explain how you obtain your answer.

(The hour hand pointed to the second minute division after 2 o'clock.)

Comment.—" This was a popular question. Less than half the candidates obtained the correct answer, 2·24, and nearly half gave 2·30. There were also many answers near to 2·24, *e.g.* 2·20, 2·25, given by candidates who stated that the time was not quite 2.30, but who made no attempt to *calculate* the exact time. The answer 2·12 was common."

III. A question about using a tablet of soap was worked in an advertisement as shown below. Write out the question of which this is the working. Then work it as you think it ought to be done.

1917

January	31 *days*
February	28 ,,
March	31 ,,
	90 *into* $4\frac{1}{2}d.$

$$\frac{9 \times 1}{2 \times 90} = \frac{9}{180} \text{ of } 1d.$$

Comment.—" Only a minority attempted this question. The answers to the first part were varied, most of them being poor to moderate. Only a few candidates introduced all the points in their question, *e.g.* date 1917, months, etc. One of the best answers was: 'A person bought a tablet of soap on Jan. 1st, 1917, which cost $4\frac{1}{2}d.$ and it lasted until Mar. 31st, how much is this per day (both days inclusive).' One solution was terse enough to be worthy of an advertising agent: 'Use Pears' Soap. Only $4\frac{1}{2}d.$ a tablet. Lasts 90 days. Costs $\frac{1}{20}$ of a penny a day.' "

The second part was moderately well done. Some candidates, apparently deceived by the simplicity of the question, attempted fanciful solutions. Most candidates performed the cancelling at the proper stage, but followed too closely the given solution. A few well reasoned solutions were received.

The general conclusions arrived at as a result of these experiments were

(1) It is possible to frame " selective " examples suited to the capacity of children of elementary school age.

(2) Children who do well at ordinary Arithmetic usually show the same ability in solving these selective problems.

(3) Examples of this type are more suited than the traditional problem to many children.

It was found that some candidates had no difficulty with the selective type although they did poorly in the rest of the paper.

Cam. Univ. Training Coll. R. S. WILLIAMSON.

GLEANINGS FAR AND NEAR.

79. In his early days Bishop Blomfield took pupils. On one occasion, when they were struggling with Euclid, Sheridan asked him: " Pray, Sir, was Euclid a good man ? " " What do you mean ? " said the tutor. " I mean was he a good, honourable, truthful person ? " " Oh, yes ! " said Blomfield, " I never heard anything to the contrary." " Then, Sir," replied the pupil, " don't you think we might take his word for this proposition ? " —[Blomfield (1786-1857); Richard Brinsley (1751-1816); Charles Francis, his brother (1750-1806). None of the famous Sheridans fits in with the story.]

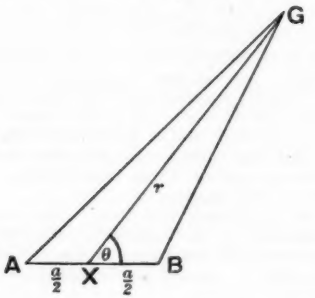
MATHEMATICAL NOTES.

581. [S. 6.] *The Sound-Ranging Problem.*

In his surmise that a method of gun-location by drawing circles "should prove rapid, and of greater accuracy than any approximation where the hyperbola is replaced by an asymptote," I think that Sir G. Greenhill is underrating the ease and accuracy of the asymptote-method actually used.

In assuming three microphones only, he is also underrating the difficulties of the method in which circles are drawn with the microphone-stations as centres. Six microphones are used in practice; and the labour of drawing six circles, and increasing their radii until they all cut, would be a formidable job even if there were no possibility of mistaking the many odd noises of a battle-front for the report of the gun in question. As a dozen wrong sounds have often to be tried and rejected before the right one is found, circle-plotting would be intolerable in the rough-and-tumble of real war.

If A and B are two consecutive microphones at a disance a apart, and G an enemy gun at a distance r from X, the middle point of AB, in a direction



making an angle θ with AB, then the sound-ranging apparatus, by recording the moments of arrival of the sound at A and B, gives us $GA - GB$, which

$$= \sqrt{r^2 + ar\cos\theta + \frac{a^2}{4}} - \sqrt{r^2 - ar\cos\theta + \frac{a^2}{4}}.$$

This "interval" being known (call it c),

$$\sqrt{r^2 + ar\cos\theta + \frac{a^2}{4}} - \sqrt{r^2 - ar\cos\theta + \frac{a^2}{4}} = c$$

can be solved as an equation in $\cos\theta$, whence

$$a\cos\theta = c\sqrt{1 + \frac{a^2 - c^2}{4r^2}}.$$

r being much larger than a and c, c is a good first approximation to $a\cos\theta$. The remarkable accuracy of this first approximation, and the rapidity with which a small correction (the "asymptote-correction") can be applied to it so as to get exact results, deserve especial notice.

The graduation of the edge of a map, mounted on a board, so that in two seconds or less a string attached to X may be laid down along the line XG (or rather an approximation to it, got by taking $a\cos\theta$ as c) can be so easily thought out that I will not use up your space by describing it. The inter-

section of five such strings gives a good first approximation to the gun's position—so good, in fact, that it is often used uncorrected.

r being now known, and values of the asymptote-correction

$$c\left(\sqrt{1+\frac{a^2-c^2}{4r^2}}-1\right)$$

having been tabulated in advance for all likely values of c and r, this correction can then be added to the observed value of c so as to give $c\sqrt{1+\frac{a^2-c^2}{4r^2}}$, the exact value of $a\cos\theta$. The strings are slightly moved to the exact directions (the work of a few seconds only), the intersection noted, and "That's him."

As an instance of the sort of size to which asymptote-corrections commonly run, take the case $a=350$ (the unit of length is always 3·376 m., the distance travelled by sound in 0·01 sec.), $c=100$, $r=2000$.

Then $\frac{a^2-c^2}{4r^2}=0{\cdot}0070$; and $c\left(\sqrt{1+\frac{a^2-c^2}{4r^2}}-1\right)$, which can always be taken as $\frac{c(a^2-c^2)}{8r^2}$ except for unusually small values of r, is only 0·35 of a unit, the unit being 0·01 of a second. From this it is evident that even when the asymptote-correction is not itself negligible, errors in it due to imperfect knowledge of r before its application are far below the limits of measurement, readings being commonly taken correct to the nearest half-unit, and only under exceptionally good conditions to the nearest tenth.

In all cases the corrections for wind and temperature are applied before the asymptote-correction. The wind-correction is of some interest, being independent of the position of the gun. W. HOPE-JONES.

582. [V. 2. 9.] The following sayings of Napoleon are quoted from R. M. Johnston, *The Corsican*:

Oct. 29, 1784 [at Brienne]. "... every one said of me, 'That boy is no good except at Geometry'" (p. 3).

Aug. 7, 1792. "I have been working a good deal at astronomy during my stay here [in Paris]. It is a splendid instrument and a superb branch of science. With my knowledge of mathematics it required very little effort to learn. It is a great acquisition" (p. 9).

June, 1817 [speaking of his memory]. "As a boy I knew the logarithms of thirty or fifty numbers" (p. 494).

Jan. 29, 1818. "To be a good general, a man must know mathematics: it is of daily help in strengthening one's ideas. Perhaps I owe my success to my mathematical conceptions" (p. 499).

Can any one give references to the French sources from which these sayings are quoted? Johnston gives no references. C. G.

583. [V. 1.] *The Census and the Teaching of Mathematics.*

The filling up of the decennial Census forms is an opportunity which must not be lost to us for teaching purposes, and the mathematical lessons deduced from it are as important as the civic. We may take it that the forms will include age, sex, birthplace, *état civile*, occupation, infirmity (if any), number of children, and number in each house. The chief points for us seem to be the following:

(1) *The idea of number.* It is important that some appreciation of the numbers involved should be obtained. One of the most effective ways of doing this is by letting the population file past at the rate of, say, one per second; massing the people in a solid square conceals the hugeness of the

magnitudes involved. From the chief quantities concerned (population, number of houses, etc.), together with such other measurements as the area of the country, the National Debt, the annual imports and exports, we can get examples of amount per unit and similar averages, and conversely from the notion of rations we get ideas of the total amount of goods consumed per year.

(2) *The idea of approximation.* The exact determination of the total population is very difficult, as we have to try to count all the tramps, folk on ships and barges, people on open spaces, sleepers-out and gipsies, people in back streets, night workers and so on. We must reckon all the babies just born, omit all the people just dead, and make certain that no one is counted twice. For the number per house we have to have some means of deciding whether a particular building is to be counted as one house, as two houses or as none at all. When we come to deal with the number of blind we shall find that there is a large number that may be omitted because their relations do not like them to be described as suffering from any infirmity. In all these cases the final figures are, despite all care, liable to possible and in some cases considerable error; the last figures in any number are less reliable than those of greater place value. The millions are correct, the tens may be quite unreliable, and therefore misleading.

(3) *The idea of functionality.* The total population changes each Census. When plotted, the curve connecting the population and the year may be smooth or may fluctuate more or less irregularly; in either case the population depends upon—is a function of—the epoch. In the same way, the population of any particular area of any class is a function of the time. The distribution of the population by age is another example, the frequency being a function of the age. Either of the two variables in any case may be taken as the independent one and the other as the dependent. Note that in all cases here, as in most physical examples, the function is continuous between the two variables; it is often singled-valued and monotone.

(4) *Statistical ideas.* In the case of the age distribution curve the smoothest curve does not pass through all the points: there may be exceptional deviations due to the loss of young adults by war casualties or by the influenza epidemic, but even when we have allowed for these we do not get a precise coincidence between the points and the smoothest curve. The deviations are due to many slight causes and are distributed themselves according to a definite law, great deviations being rarer than small ones. The deviations must be removed if the curve is to be of any value to insurance offices to give the chance of death and the expectation of life: to a smaller degree the corrections must be applied in forecasting the likely number of Old Age Pensioners, of new houses required for newly married couples and for the probable number of children who will require schooling in a year's time. Other cases of deviation from the mean are exhibited by the variation between local and national values, in such cases as the sex ratio at various ages and at birth, the ratio of infants to adults, of married to single. The general stability of such ratios, of the birth rate, and the marriage and the death rate, all demonstrate that though each individual case has reasons peculiar to itself, yet in the mass quite precise forecasts can be made of their behaviour, just as we can deal with the properties of a gas as a whole though we do not and cannot know the individual velocities of the separate molecules.

We have the idea of correlation by consideration of the relation of the age of husband to that of wife, of longevity to *état civile*, of occupation to residence, in all of which cases we do not have the absolute dependence of the physicist, but merely have a general tendency of one quantity to vary as the other.

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REVIEWS.

Higher Mechanics. By HORACE LAMB. 25s. net. pp. 270. 1920. (Cambridge University Press.)

Lagrange and Laplace, of Napoleonic traditions, had given place to the school of Whewell, Parkinson, Todhunter, Routh, of the mid-Victorian era; and their analytical treatment is subjected in their turn to the scrutiny of this century, in which the methods employed by them have been overhauled, to render obsolete the bulk of Victorian tradition.

The approximations of their differential equations can be replaced by an exact geometrical treatment, in which ideas take the place of symbols and intelligible propositions supersede equations.

"Examiner les choses en elles-mêmes," as Poinsot recommends, "sans recours aux équations analytiques"; "dans toutes ces solutions (de Lagrange) on ne voit guère que des Calculs, sans aucune image nette de la rotation du corps."

Take the mid-Victorian treatment in Routh of deflection in artillery fire due to diurnal rotation, with its otiose superfluity of data. Because only the time of flight requires to be stated. The vertical plane of fire will continue to follow a rising or setting star, at a horary rate of 15° sin. lat. in azimuth, that is 15′ sin. lat. per minute, or 0′·25 sin. lat. per second; the vertical plane remains fixed while the Earth turns round underneath.

In the bombardment of Paris by the German giant gun, a time of flight of 3 minutes was estimated, and this in latitude 50° makes the deflection about 34′, or one in 100; say 1·2 kilometres in a range of 120 km. But the drift from the vertical plane would wipe out this deflection ten-fold.

In a French caricature the Germans had brought a 100 mile giant gun up to Calais, for the bombardment of London. The German officer is shown looking over the gun through his binocular, exclaiming "Donnerwetter! missed London." But he does not see that the shell has passed round the world and is taking him in rear.

Allowing for the rotation of the Earth, we infer that the shot was fired the day before, 24 hours earlier, and is completing the seventeenth circuit, as a Puck or grazing satellite.

In practice any deviation due to diurnal rotation could never be disentangled from the effect, far more important, of drift from the vertical plane of fire of an elongated projectile fired from rifled artillery; so that these calculations are otiose, except for a discussion of the deviation from the plumb line of a body falling down a deep shaft, or of a bomb from the vertical dropped from a height in the air.

The Rigid Dynamics in three dimensions is concerned chiefly with gyroscopic theory, and the new methods of this century can be illustrated in their treatment of this problem.

Take a top spinning nearly upright at a constant inclination OC', with the point placed in a smooth cup O in Fig. 1, to prevent it from wandering about over the table; and so we realise the condition called perfectly rough in the hideous jargon of the text book. So too for the motion of a top with a sharp point on a smooth table realised with a ball point in a socket of ball bearings.

There is little need to recall the notation in general use; and so if θ is the inclination of the axle to the upright, zenith distance, and the vector OK represents the axis of resultant angular momentum, the velocity of K is to be equated to the impressed couple of the preponderance Mh; or on the figure 1:

$$\mu . CK = gMh \sin\theta, \quad OM = gMh/\mu,$$

the dynamical condition, with constant precession μ.

Draw KC' perpendicular to the axle; then OC' represents the angular momentum CR of the spin R; and $KC' = A\mu \sin\theta$, $C'M = A\mu\cos\theta$, the kinematical condition. Then in the geometrical condition

$$C'M + MO = C'O, \quad A\mu\cos\theta + \frac{gMh}{\mu} = CR,$$

the condition of steady motion, a quadratic for μ, and the solution is shown graphically by drawing a straight line parallel to OC' to cut the circle on OC' in two points Q, Q', with ordinates QM, $Q'M'$, and $QM^2 = gMhA\cos\theta$, to scale.

Hanging down inert, vertically from O, the top forms a compound pendulum of S.E.P. (simple equivalent pendulum length) $OP = l = A/Mh$, and will make n/π single beats per second in invisible oscillation where $n^2 l = g$, $An^2 = gMh$; and the S.E.P. l is used in the manner of Galileo, instead of a timepiece, to beat the equal intervals of time taken as the unit.

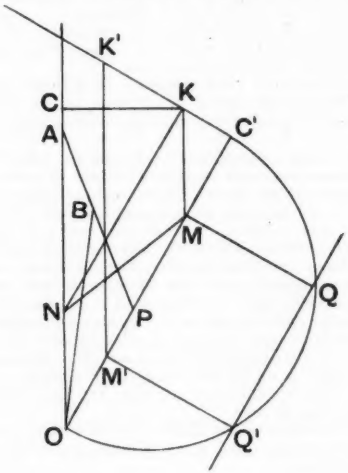

FIG. 1.

Spinning upright, with $\cos\theta = 1$, and $\mu + \frac{n^2}{\mu} = \frac{C}{A} R$, the two values of μ will give the independent normal invisible circling of the axle round the vertical; and the general slight oscillation will be compounded of these two circlings, giving an epicycle motion to the axle, as in the ancient Ptolemaic astronomy. And so we arrive at this result by an exact geometrical treatment, without recourse to the mid-Victorian method of approximation through simultaneous differential equations of rectilinear motion; and there is no place for a Berkeley to pick a hole in the argument.

The difficulties of straight line vibration and the oscillation of the plane pendulum are removed by the greater simplicity, analytical and mechanical, of the conical pendulum and uniform circling motion. A new Dynamic can be written in this the order recommended by Aristotle.

The discussion of the slight oscillation of the axle in the upright position is of importance in the practical problem of a separating centrifugal machine, actuated from below, where the upper spindle must be free to secure smoothness of running.

Incidentally, by means of Kirchhoff's Kinetic Analogue, the stability is investigated, in the same exact elementary method, of a long straight shaft between bearings, transmitting a wrench, of a thrust and couple. And the spin can be calculated as required for the axial stability of an elongated bullet, when the upsetting couple is assigned of the air medium, to replace the gravity couple.

With the axle at a constant inclination θ, and a rapid spin, making OC' great compared with MQ on the figure 1, one root of the quadratic for μ is small, and given by $\frac{n^2}{\mu}=\frac{CR}{A}$, $\mu=\frac{An^2}{CR}$, and this is the precession obtained in the Kindergarten treatment, where the deviation of OK from OC' is treated as insensible, and the velocity of C' is equated to the gravity couple.

Then $\frac{n}{\mu}=\frac{C}{A}\cdot\frac{R}{n}$, or expressed in words, the number of single beats per circuit of the axle round the vertical is C/A times the number of revolutions, per double beat, of S.E.P. *l*.

The large root of the quadratic is given by $\mu=\frac{C}{A}\,R\sec\theta$, and here OK comes into coincidence with the vertical OC; the precession changes sign in passing through infinity when the axle is horizontal, and with the axle pointing downward this precession is retrograde and violent, not like the former easy smooth motion; and Kelvin's rule is reversed with regard to the effect of hurrying the precession.

A simple apparatus for the experimental verification is provided by a bicycle wheel with alt-azimuth suspension, and spun by hand.

The gyroscope of the frontispiece of *Higher Mechanics* has an unfamiliar appearance; there is no indication of how the flywheel is set spinning. The treacherous use of string is to be avoided in a lecture for a variety of practical reasons, such as the risk of entanglement, or dragging the axle out of the pivots.

Elliott's Gyro-Unit is actuated by electricity, up to gyrocompass speed, and more suitable for the lecture table. But an ordinary bicycle wheel makes a very cheap efficient substitute; the axle in its ball bearings is prolonged both ways by a stalk and to a point O, which can be placed in a cup, to keep it fixed; and the wheel is then spun by hand.

Grasped by the stalk, the gyroscopic effect is felt when it is brandished; and the end of the stalk may rest on the hand, and the wheel left to its own precession while the body is wheeled round; and Kelvin's effect then investigated of hurrying or delaying the axial precession.

Although the precession of the axle and its inclination θ may appear perfectly steady, it is always accompanied by a slight invisible trepidation or nutation, of finite period and apsidal angle, as we see realised with a plummet at the end of a thread, either in invisible oscillation hanging down, or else whirled round as a conical pendulum; a progressive motion of an apse is always observable, as employed by Newton in illustrating the motion of the Moon's perigee.

In the investigation a change is made to a penultimate state of visible unsteady motion; and then the vector of angular momentum changes from OK to OH, with $KH=A\frac{d\theta}{dt}$, the component perpendicular to the plane COC'; and the velocity of H is equated to the gravity couple, $An^2\sin\theta$, and in the direction KH; so that

$$(1)\qquad An^2\sin\theta=\frac{d\,.\,OH}{dt}\cdot\frac{OH}{KH},\quad A^2n^2\sin\theta\frac{d\theta}{dt}=\frac{d\frac{1}{2}OH^2}{dt},$$

and integrating, $$(2)\qquad OH^2=2A^2n^2(F-\cos\theta),$$

where F is an algebraical constant of the motion; this relation can be interpreted also as a form of the Equation of Energy.

Then since, (3) $HK^2=OH^2-OK^2$,

$$OK^2\sin^2\theta=CC'^2=OC^2-2OC\,.\,OC'\cos\theta+OC'^2,$$

$$(4)\qquad A^2\left(\frac{d\theta}{dt}\right)^2=2A^2n^2(F-\cos\theta)-\frac{CC'^2}{\sin^2\theta},\text{ or with }\cos\theta=z.$$

$$(5)\qquad \left(\frac{dz}{dt}\right)^2=2n^2(F-z)(1-z^2)-\frac{CC'^2}{A^2}=n^2Z,$$

$$Z=2(F-z)(1-z^2)-\frac{CC'^2}{A^2n^2},$$

and this defines z as an elliptic function of t; or as pointed out by Euler, the motion of the axle in altitude θ beats time with the finite oscillation of a plane circular pendulum.

With the cubic Z resolved into linear factors,

(6) $Z=2.z_1-z.z_2-z.z-z_3$, $z_1>1>z_2>z>z_3$; and z is given as an elliptic function of t in the finite movement, in the form

$$(7)\qquad z=z_2\mathrm{sn}^2\tfrac{1}{2}mt+z_3\mathrm{cn}^2\tfrac{1}{2}mt,\quad \kappa^2=\frac{z_2-z_3}{z_1-z_3},\quad \frac{m^2}{n^2}=2(z_1-z_3).$$

The velocity of H imparted by the gravity couple is $An^2\sin\theta$ along KH; and this velocity is the rate of growth of HK added to the velocity of K carried round by the plane OCK; so that, putting $\frac{d\theta}{dt}=Q$, and $A\sin\theta\frac{d\psi}{dt}=KC'$,

$$(8)\quad An^2\sin\theta=A\frac{dQ}{dt}+CK\frac{d\psi}{dt}=A\frac{dQ}{dt}+\frac{CK.KC'}{A\sin\theta},$$

with (9) $$CK=\frac{OC'-OC\cos\theta}{\sin\theta},\quad KC'=A\sin\theta\frac{d\psi}{dt}=\frac{OC-OC'\cos\theta}{\sin\theta}.$$

$$(10)\qquad A\frac{dQ}{dt}=An^2\sin\theta-\frac{(OC'-OC\cos\theta)(OC-OC'\cos\theta)}{A\sin^3\theta},$$

obtainable also by a differentiation of (4), or else (4) is obtained by its integration.

Differentiating (8) with respect to θ, with

$$\frac{d}{d\theta}=\frac{1}{Q}\frac{d}{dt},\quad\text{and}\quad\frac{dCK}{d\theta}=\frac{KC'}{\sin\theta},\quad\frac{dKC'}{d\theta}=\frac{CK}{\sin\theta},$$

$$(11)\qquad \frac{1}{Q}\frac{d^2Q}{dt^2}=n^2\cos\theta-\frac{CK^2+KC'^2-CK.KC'.\cos\theta}{A^2\sin^2\theta}$$

$$=\left(n^2+\frac{3OM.ON}{A^2}\right)\cos\theta-\frac{OK^2}{A^2}.$$

In a state of Steady Motion, θ is constant, Q and $\frac{dQ}{dt}$ are zero, apparently,

$$(12)\qquad OM.ON=A^2n^2,\quad\frac{OM}{An}=\frac{n}{\mu},\quad\frac{ON}{An}=\frac{\mu}{n},\quad\frac{OM}{ON}=\frac{n^2}{\mu^2}=\frac{\lambda}{l},$$

with λ the height of the equivalent conical pendulum,

$$\lambda=g/\mu^2=OA,\quad l=g/n^2=OP,$$

so that OM, ON are inversely as l, λ or OP, OA;

$$(13)\qquad\frac{OC}{An}=\frac{n}{\mu}\cos\theta+\frac{\mu}{n},\quad\frac{OC'}{An}=\frac{n}{\mu}+\frac{\mu}{n}\cos\theta;$$

$$(14)\qquad\frac{CC'^2}{A^2n^2}=\frac{OK^2\sin^2\theta}{A^2n^2}=\left(\frac{n^2}{\mu^2}+2\cos\theta+\frac{\mu^2}{n^2}\right)\sin^2\theta.$$

Then in the slightly tremulous nutation, making m/π beats per second.

$$(15)\qquad\frac{1}{Q}\frac{d^2Q}{dt^2}=-m^2,\quad Q=Q_0\cos(mt+\epsilon)=Q_1\cos mt+Q_2\sin mt,$$

$$(16)\qquad\frac{m^2}{n^2}=\frac{OK^2}{A^2n^2}-4\cos\theta$$

$$=\frac{\lambda}{l}-2\cos\theta+\frac{l}{\lambda}=\frac{AP^2}{OA.OP}=2(z_1-z_3).$$

$$(17)\qquad z_1=\tfrac{1}{2}\left(\frac{\lambda}{l}+\frac{l}{\lambda}\right)=\frac{OA^2+OP^2}{2OA.OP},\quad z_1,\ z_3=\frac{OB^2\pm AB^2}{OA.OP},$$

with B the mid-point of AP.

Here z_2, z_3 close in on z, and we can replace $z_1 - z$ in (6) by $z_1 - z_3$, and write

$$(18) \qquad \left(\frac{dz}{dt}\right)^2 = 2n^2(z_1-z_3)(z_2-z)(z-z_3) = m^2(z_2-z)(z-z_3),$$

$$z = z_2 \sin^2 \tfrac{1}{2}mt + z_3 \cos^2 \tfrac{1}{2}mt;$$

degenerate case of (7) with $\kappa = 0$, the axle in apparently steady motion making m invisible nutations per n invisible beats of pendulum length l.

With Ψ the apsidal angle, the nutations per circuit of the axle

$$(19) \qquad \frac{2\Psi}{2\pi} = \frac{\mu}{m} = \frac{\mu}{n} \cdot \frac{n}{m} = \sqrt{\frac{OP}{OA}} \cdot \frac{\sqrt{(OA \,.\, OP)}}{AP} = \frac{OP}{AP}.$$

Realised with a spherical pendulum, a plummet G at the end of a thread OP, making an angle with the downward vertical nadir, OK is at right angles to OP, $\frac{n}{\mu} - \frac{\mu}{n}\cos\theta = 0$, $\lambda = l\cos\theta$;

$$(20) \qquad \frac{\Psi}{\pi} = \frac{OP}{AP} = \frac{1}{\sqrt{(4\cos^2\theta + \sin^2\theta)}} = \frac{1}{\sqrt{(4 - 3\sin^2\theta)}}, \quad \frac{\Psi}{\frac{1}{2}\pi} \to 1 + \tfrac{3}{8}\sin^2\theta,$$

when θ is small, Lagrange's result, corrected by Bravais.

But Ψ is always present, and $> \frac{1}{2}\pi$, although the nutation is invisible.

The word approximate has so far not been required in this treatment.

Spinning upright, with $CR > 2An$, the top is stable. But with $CR < 2An$, $h = \dfrac{CR}{2A} < n$, the upright position is unstable; and with $CR = 2An\cos\frac{1}{2}\theta_3$, $h = n\cos\frac{1}{2}\theta_3$, the axle of the top sinks to an inclination $\theta = \theta_3$ with the upright, but in an infinite time.

After that the top rises again, on the law of secular stability,

$$(21) \qquad \sin\tfrac{1}{2}\theta = \sin\tfrac{1}{2}\theta_3 \operatorname{sech}\tfrac{1}{2}mt, \quad m = 2n\sin\tfrac{1}{2}\theta_3, \text{ or}$$

$$(22) \qquad \psi - nt\cos\tfrac{1}{2}\theta_3 = \cos^{-1}\frac{\cos\frac{1}{2}\theta_3}{\cos\frac{1}{2}\theta} = \tan^{-1}(\tan\tfrac{1}{2}\theta_3 \operatorname{th}\tfrac{1}{2}mt),$$

a relation connecting true anomaly $2\psi - mt\cot\frac{1}{2}\theta_3$ with hyperbolic excentric anomaly mt in an hyperbolic orbit of excentricity $\sec\theta_3$, about a focus external to the branch.

The azimuth ψ can be given also by the sum of a circular and hyperbolic sector, swept out by the same central radius vector.

The relation between t, θ and $\psi - ht$ can also be shown on a spherical triangle, and the Mercator chart. Examine the special case of $\theta_3 = \frac{1}{2}\pi$.

Although this top returns eventually to its original upright position, it takes eternity to do so, and is infinitely sluggish; so that the motion can hardly be called stable any more than if it was a plane pendulum balanced originally upright. The addition seems required to the usual definition of dynamical motion, as stable or unstable, of the words—in a finite time—as in the lemmas of the Principia, unless the idea is introduced of Secular Stability.

To gain an idea of the nature of the motion when the upright position is on the verge of instability, and the axle is beginning to break away from the vertical, take $\theta_3 \to 0$; QQ' in fig. 1 is grazing the circle on OC', and we can take $CR = 2An$, $h = n$, $\psi = nt$.

Then with $\dfrac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta_3} = \dfrac{\rho}{a}$, the curve described by the axle can be written

$$\rho \operatorname{ch}(\psi \sin\tfrac{1}{2}\theta_3) = a,$$

a Cotes's spiral.

In the general case of unsteady motion again, the determination of the curve of H, as well as of the azimuth ψ round the vertical, requires an elliptic integral of the third kind; and another one is required for ϕ, the hour angle of the motion with respect to the axle; but the motion in altitude θ, given in (7), beats time with a simple pendulum in finite oscillation, as Euler pointed out, 1748.

A spherical ball, spinning and rolling on a spherical surface, either inside or outside, will keep step with an associated gyroscopic top motion. When the flywheel has no spin, or the ball either about the common normal of the spheres, the motion is comparable with a spherical pendulum motion realised with a plummet or a thread and defined by its S.E.P. l as much as in plane oscillation; but no appreciable simplification is made in the analysis; on the contrary the condition of no spin introduces a cramping condition not easy to realise in a numerical case, as seen in (45).

Proceed next in steady gyroscopic motion with a gyroscope swung round at a steady inclination θ with the downward vertical at the end of a thread DE at an inclination ϕ, attached to the axle EG at E and to a fixed point at D, as in fig. 2.

When no spin is given to the gyroscopic flywheel, the geometrical conditions shown in fig. 2 will soon be found by the careful diligent student, where

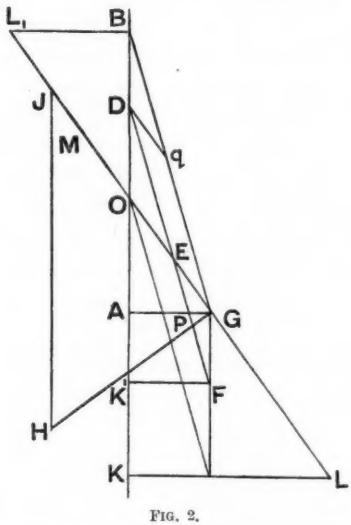

FIG. 2.

$AB = OK = DK' = \lambda = g/\mu^2$, the height of the equivalent conical pendulum; then (23) $OE \,.\, EF = (\lambda \sec\theta - EL)(\lambda \sec\phi - DE) = DE \,.\, EG$, where $EG \,.\, GL = A/M$, the quadratic equation for the two values of λ.

With θ and ϕ small the normal oscillations are obtained of the double pendulum, thread DE and body EGL, as the shadow of the conical revolution, where the two parts swing together, either to one side and the other, or else swing across each other, with the condition (24) $(\lambda - DE)(\lambda - EL) = DE \,.\, EG$.

But suppose this condition is not satisfied; the system can be trimmed by giving the flywheel a spin R and angular momentum $K = CR$, represented geometrically by JM, with $GH = A\mu \sin\theta$, $HJ = A\mu$, $GJ = A\mu\cos\theta$.

The moment about G of the tension of the thread, $gM \sec\phi$, is

$$(25) \qquad gM \,.\, GP, \quad \text{with} \quad \frac{GP}{GE} = \frac{BL_1}{GL_1} = \frac{OL_1}{GL_1}\sin\theta = \frac{OB}{AB}\sin\theta;$$

so that, with $g = AB \,.\, \mu^2$, the moment is $M\mu^2 \,.\, OB \,.\, GE \sin\theta$; and equating on the dynamical principle this moment to the velocity of the angular momentum vector,

(26) $M\mu^2 . OB . GE = A\mu^2 \sin\theta\cos\theta - K\mu\sin\theta,$

$$\frac{JM}{JG} = \frac{K}{A\mu\cos\theta} = 1 - \frac{M . OB . GE}{A\cos\theta} = 1 - \frac{M . OL_1 . GE}{A} = 1 - \frac{OL_1}{GL},$$

with $$\frac{A}{M} = GE . GL, \quad 1 - \frac{JM}{JG} = \frac{OL_1}{GL} = \frac{OB}{AK}.$$

With no spin, M coincides with J, and $OB = AK$, the condition stated previously.

Expressed analytically with $GE = a$, $ED = b$,

$$GL = \frac{A}{Ma}, \quad EL = a + \frac{A}{Ma} = l, \quad AB = \lambda,$$

$$\frac{OA}{\lambda} = \frac{OG\cos\theta}{GB\cos\phi} = \frac{Dq}{Bq}\frac{\cos\theta}{\cos\phi} = \frac{a\cos\theta}{\lambda - b\cos\phi},$$

(27)
$$\begin{aligned}\frac{K}{A\mu} &= \cos\theta - \frac{OB}{GL} = \cos\theta - \frac{\lambda}{GL} + \frac{OA}{GL} \\ &= \cos\theta - \frac{\lambda}{GL} + \frac{\lambda\cos\theta}{GL(\lambda - b\cos\phi)} \\ &= \cos\theta - \frac{Ma\lambda}{A} + \frac{Ma^2\lambda\cos\theta}{A(\lambda - b\cos\phi)}, \text{ with } \lambda = \frac{g}{\mu^2};\end{aligned}$$

a quartic equation for μ, with given θ and ϕ, similar to those encountered in the discussion of the stability of a flying machine. A root of (27) is read off at the intersection of a parabola and hyperbola.

With $\theta = 0$, $\phi = \pi$, the condition is obtained for a top spinning in a smooth spherical cup or saucer in the neighbourhood of the lowest point; and the four values of μ determine the angular velocity of the four arms of arbitrary length, giving the motion of an epicycle, and so assign the small oscillation, without recourse to an approximation in a differential equation.

So too with $\theta = \pi$, $\phi = \pi$, for a top spinning near the summit of a smooth sphere, or balanced on the tip of a light rod.

This rod may be supposed to carry a flywheel, and the conditions obtained of the "top or the top of a top," realised with one bicycle wheel spinning in a cup on the tip of the stalk of the lower wheel, two links of a gyrostatic chain, as in the Chu Chin Chow poster.

The same figure 2 will serve for a round body rolling steadily on a table, if DE produced cuts the surface at the point of contact; and then the reaction of the table replaces the tension of the thread.

The heavy analytical calculations of a mid-Victorian Routh are replaced in this new method by analytical measurement on a drawing, and no approximation is required.

Similar problems of the small vibration of a pliable system, such as a string of beads on a taut cord, or of a chain hanging vertically are best treated by this reduction to the equivalent steady motion of bodily rotation about the axis as no approximation need be introduced; a plane normal vibration can then be treated as the shadow on a wall.

The terrific approximations employed by Sir William Thomson in his discussion of the gyrostatic chain can thus be replaced by an exact geometrical treatment on the previous elementary methods, by supposing the chain to form a regular polygonal helix whirled round a central axis bodily; no restriction is needed to small displacement until the solution is complete. The idea is useful as a mechanical picture of the rotary polarisation of light; and then it can also be proved that a gyrostatic chain can be made to stand upright, as in the reported rope trick of the Hindoo conjurer.

In all these discussions some simple homely apparatus should be at hand, to fix the ideas; a simple pendulum of a thread and plummet, some length of chain or beads, a small top on the table, or else a bicycle wheel, or a Maxwell top.

Why not point out here on p. 248 that Southwell has arrived at Legendre's differential equation for his zonal harmonic? and turn the reader over to the

usual place in the book ? Many another example would be better for editing from the original form in which it was set.

The purely abstract treatment is extinct in this century of the school of Lagrange and Laplace, a procession of formulas with no appeal to experiment. Who reads the *Mécanique céleste* to-day ? The reader never arrives: he is compelled to alight and walk a long way to reach his journey's end.

Lagrange's equations of motion have become a stock piece of cram, to be expected as confidently as the binomial theorem and the common pump in former days. But the idea is beyond the grasp of any but a select few, and it proves that Lagrange himself did not understand them fully, unaware as he was of the distinction of coordinates as holonomic and anholonomic (lovely pulp words !). The difficulty seemed to arise first when Ferrers tried to apply Lagrange's equations direct to the stability of a hoop ; here the wave length of a sinuosity of the track is a more eloquent picture than the time of vibration.

The application of Lagrange's equations to the general motion of the unsymmetrical top has not led to any advance ; this is the intractable problem of to-day in gyroscopic dynamics. It can be shown off experimentally on the table in front by Maxwell's dynamical top, with the screws set so as to throw the bell as much as possible out of dynamic balance, spun in motion resting in the agate cup O on the pillar by a twirl of the fingers.

Visualised in this way, the conditions of the motion may be stated in a couple of propositions :

I. The vector of angular momentum, OH, describes a curve in a horizontal plane, and the projection of G, the centre of gravity, describes a hodograph of the curve of H.

The determination of this curve of H is an intractable problem analytically, except in a few isolated cases, when the body is unsymmetrical.

II. But in the usual symmetrical top, of uniaxial symmetry, the curve of H is a Poinsot herpolhode, described by rolling an ellipsoid fixed in the body, with centre at O, on the fixed horizontal plane through H.

The solution is then reduced to this problem of Kinematical Geometry.

These are the two intrinsic relations of gyroscopic geometry ; and the reduction to the case where the curve of H can be made algebraic, and the associated top motion as well, is an interesting application of Abel's theory ; a collection of such results makes a valuable feature of the applications better than any amount of arid generalities.

In the general case the determination of the curve of H, as well as of the azimuth ψ round the vertical of the axle of the top, and of ϕ the hour angle round the axle, require the third elliptic integral in all its generality ; but the motion in altitude θ beats time, as shown already in (7), with simple pendulum motion in finite oscillation ; but the third elliptic integral was beyond the powers of Euler.

The symmetrical uniaxial spinning top may then be said to speak the language of elliptic function theory, as if it was created expressly for the purpose.

With (ρ, ϖ) the polar coordinates of H in its horizontal plane,

$$(28) \qquad \rho^2 = CH^2 = 2A^2n^2(E - \cos\theta), \quad E = F - \frac{OC^2}{2A^2n^2}.$$

$$(29) \qquad \rho\frac{d\varpi}{d\rho} = \tan CHK = \frac{CK}{HK} = \frac{CK\sin\theta}{HK\sin\theta} = \frac{OC' - OC\cos\theta}{An\sqrt{(Z)}},$$

and here the third elliptic integral for ϖ appears, when $\cos\theta$ is expressed by ρ^2.

Also $\psi = \varpi - HCK = \varpi + CHK - \frac{1}{2}\pi$, so that azimuth ψ depends on the same third elliptic integral. But another integral is required by the hour angle ϕ.

To a geometrical scale, with

$$(30) \qquad \frac{CH,\ OC,\ OC'}{2An} = \frac{\rho,\ \partial,\ \partial'}{k}, \quad E - z = 2\frac{\rho^2}{k^2},$$

(31) $$Z=2(E-z)(1-z^2)-\frac{(OC'-OC\,.\,z)^2}{A^2n^2}$$

$$=4\frac{\rho^2}{k^2}\left[1-\left(E-2\frac{\rho^2}{k^2}\right)^2\right]-4\left(\frac{\partial'-\partial E+2\partial\frac{\rho^2}{k^2}}{k}\right)^2$$

$$=4\frac{R}{k^6},$$

(32) $$R=4\,.\,\rho_3{}^2-\rho^2\,.\,\rho^2-\rho_2{}^2\,.\,\rho^2-\rho_1{}^2,\quad Z=2\,.\,z_1-z\,.\,z_2-z\,.\,z-z_3,$$

$$z_1>1>z_2>z>z_3,\quad \rho_3{}^2>\rho^2>\rho_2{}^2>0>\rho_1{}^2.$$

(33) $$nt=\int_{z_3}^{z}\frac{dz}{\sqrt{(Z)}}=\int_{\rho}^{\rho_3 k}\frac{d\rho^2}{\sqrt{R}},\qquad ht=\int\frac{\partial\,d\rho^2}{\sqrt{R}},$$

(34) $$\varpi=\int\frac{2\partial\rho^2+(\partial'-\partial E)\,k^2}{2\rho^2}\,\frac{d\rho^2}{\sqrt{R}},$$

and with $$\rho=0,\quad R_0=-(\partial'-\partial E)^2\,k^4=4\rho_1{}^2\rho_2{}^2\rho_3{}^2.$$

(35) $$\varpi=\int\frac{\partial\,d\rho^2}{\sqrt{R}}+\int\frac{\frac{1}{2}\sqrt{(-R_0)}}{\sqrt{R}}\,\frac{d\rho^2}{\rho^2},$$

in a standard form of Weierstrass, to which Abel's theory can be applied for the construction of the algebraical cases.

In the $(p,\ \rho)$ relation of the herpolhode of H,

(36) $$p^2=\rho^2\sin^2CHK=\frac{[2\partial\rho^2+(\partial'-\partial E)\,k^2]^2}{2\rho^2+(1-E)\,k^2\,.\,-2\rho^2+(1+E)\,k^2}.$$

(37) $$\frac{p^2+\partial^2-\partial'^2}{k^2}=\left[\frac{2\partial'\rho^2+(\partial-\partial'E)\,k^2}{2\partial\rho^2+(\partial'-\partial E)\,k^2}\right]^2,$$

$$\frac{OK^2-OC'^2}{OK^2-OC^2}=\left[\frac{2\partial'\,.\,OH^2+(\partial-\partial'F)\,k^2}{2\partial\,.\,OH^2+(\partial'-\partial F)\,k^2}\right]^2.$$

Then Prop. I, that the projection of the C.G. of the top describes a hodograph of the curve of H, is expressed analytically by

(38) $$\frac{d}{dnt}\left(\frac{\rho}{k}\exp\varpi i\right)=\tfrac{1}{2}\sin\theta\exp(\psi+\tfrac{1}{2}\pi)\,i.$$

To illustrate Abel's theorem, and to add some actual numerical interest to an abstract treatment, bisect a period for the parameter of the third elliptic integral. A secular term can be cancelled by giving the top or flywheel an appropriate spin (not possible however with the non-spinning gyroscope or equivalent spherical pendulum); and then the herpolhode equation of H takes the form, in four branches,

(39) $$\rho^2\exp 2\varpi i=\sqrt{(\rho^4+k^2\rho^3-k^2\rho_3{}^2)}+ik\sqrt{(\rho_3{}^2-\rho^2)}.$$

When the period is trisected,

(40) $$\rho^3\exp 3\varpi i=(\rho^2-2Ak^2)\sqrt{(\rho^2-\rho_1{}^2)}+iAk\sqrt{(\rho_3{}^2-\rho^2\,.\,\rho^2-\rho_2{}^2)},$$

and expressed in terms of a parameter b,

$$3\frac{\partial}{k}=A=\tfrac{1}{2}(b^2-9),$$

$$\frac{\rho_1{}^2,\ \rho_2{}^2,\ \rho_3{}^2}{k^2}=-\tfrac{1}{4}(b^2-1)(b^2-9),\quad (b+1)(b-3),\quad (b-1)(b+3).$$

This is for $b>3$, and three loops; but with $3>b>1$, ρ_1 and ρ_2 change place, and there are six loops.

Similarly for a quinquisection of a period, in terms of a parameter c, and

$$\partial=\tfrac{1}{4}(c+3)(c^2-4c-1)k,$$

(41) $\rho^5 \exp 5\varpi i$

$$= [\rho^4 - 4(2c^2+5c+1)(c^2-4c-1)k^2\rho^2 - 16c(c+1)^3(c^2-4c-1)^2k^4]$$
$$\sqrt{[\rho^2+(c+1)^3(c-1)(c^2-4c-1)k^2]}$$
$$+ i[(c+3)(c^2-4c-1)k\rho^2 + 4(c+1)^3(c^2-4c-1)^2k^3]$$
$$\sqrt{[-\rho^4 + 8c(c^3+c^2-3c-1)k^2\rho^2 - 16c^2(c+1)^3(c-1)(c^2-4c-1)k^4]}$$

In the next case the form will be

(42) $\rho^7 \exp 7\varpi i = (\rho^6 - Ak^2\rho^4 + Bk^4\rho^2 - Ck^6)\sqrt{(\rho^2-\rho_1{}^2)}$
$$+ i(Pk\rho^4 - Qk^3\rho^2 + Rk^5)\sqrt{(\rho_2{}^2-\rho^2 . \rho^2-\rho_2{}^2)}, \text{ will } \delta = \tfrac{1}{7}Pk.$$

And so on with the material given in the *Report on Gyroscopic Theory* 1914, Chap. V; but the complication increases rapidly. A differentiation, as in (38), will then give the associated motion of the axle of a top.

To construct a figure of an actual case, take some appropriate numerical value of b in (40):

1. $b = 2\sqrt{3}+3 = \tan 60^\circ \tan 75^\circ$, giving a modular angle 15°, and a top figure enclosed in an equilateral triangle, a close fit.

And $b = \sqrt{3}$ gives six cusps, and the same modular angle.

2. $b = (\sqrt[3]{2}+1)^2$, a figure complete in three cusps.

3. $b = \sqrt{3} + \sqrt{2}\sqrt[4]{3}$, and a modular angle 45°.

4. $b+1 = [\sqrt{(\sqrt[3]{2}+1)} + \sqrt{(\sqrt[3]{2}-1)}]^3$, or $\sqrt{\left(\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}\right)}$, and a modular angle 75°.

And in quinquisection of (41), $c=2$, or $-\frac{1}{2}(\sqrt[3]{10}-1)$, will give a top motion complete in ten cusps, and in five cusps with c about 7.4 ..., or -0.108.

The brachistochrone has been left untouched in this *Higher Mechanics*. But for the gravity brachistochrone on a sphere, it is worth mention that it is traced by the axle of a spinning top, held fixed and spun, and then released, so as to start from a cusp; and the curve is then rectifiable.

Starting at an angle θ_2 with the upward vertical zenith, at $\frac{A}{C}\sqrt{(\frac{1}{2}\sec\theta_2)}$ revolutions per pendulum l beat, this brachistochrone reaches down to the equator, and then rises up again; and it is expressible in a simple form:

(43) $$\cos 2(\psi - ht) = \mathrm{dn}\, 2mt = \cos\frac{s}{a}, \quad h = n\sqrt{(\tfrac{1}{2}\cos\theta_2)}$$

(44) $$\sin\theta \exp(\psi - ht) = \sqrt{(1-\cos\theta_2\cos\theta)} + i\sqrt{(\cos\theta_2\cos\theta - \cos^2\theta)}$$
$$= \mathrm{dn}\,\tfrac{1}{2}mt + i\kappa\,\mathrm{sn}\,\tfrac{1}{2}mt\,\mathrm{cn}\,\tfrac{1}{2}mt.$$

With $\theta_2 \to \frac{1}{2}\pi$, the penultimate case is obtained of a necklace round the equator of brachistochrone cycloids.

An associated case of motion may be stated here, of a spherical pendulum, giant stride, or non-spinning gyroscope, where the axle rises to the horizontal position, and sinks down again to an inclination θ_3 with the downward vertical nadir; here

(45) $$\sin\theta \exp(\psi - ht)\, i = \frac{h}{n}\sqrt{(2\cos\theta)} + i\sqrt{\left(1 - 2\frac{h^2}{n^2}\cos\theta - \cos^2\theta\right)}$$
$$= \sqrt{(\sec\theta_3 - \cos\theta_3 . \cos\theta)} + i\sqrt{(\cos\theta_3 - \cos\theta . \cos\theta + \sec\theta_3)}$$
$$= \sin\theta_3\,\mathrm{cn}\,\tfrac{1}{2}mt + i\frac{\mathrm{sn}\,\frac{1}{2}mt\,\mathrm{dn}\,\frac{1}{2}mt}{\kappa'},$$

with (46) $\sin^2\theta\frac{d\psi}{dt} = 2h$, $v_2{}^2 = 4l^2h^2 = 2gl(\sec\theta_3 - \cos\theta_3)$, $v_3{}^2 = 2gl\sec\theta_3$;

and in the conical pendulum, at constant angle θ_3, $v^2 = \frac{1}{2}v_3{}^2$.

The apsidal angle

(47) $$\Psi = \tfrac{1}{2}\pi + K\sqrt{(\kappa'^2 - \kappa^2)} = \tfrac{1}{2}\pi + K\kappa'\sin\theta_3 \to \pi\left(1 - \tfrac{3}{8}\frac{l^2}{a^2}\right)$$

when the pendulum is whirled round rapidly, apparently in a horizontal circle, as $\theta_2 \to \frac{1}{2}\pi$, and v_2 is large, $\to 2gl \sec\theta_2 = 2ga$, suppose.

But with v_2 and θ_2 small, the spherical pendulum grazes the nadir, and the modular angle becomes 45°.

Interesting exercises in elliptic function theory are provided by the various questions of a pendulum, clock, or governor, fixed in a whirling arm, and the solution should be carried out to the fullest extent, with no approximation.

Then there is the application to the effect on the compass of the vibration of a flying machine. And the flight of the machine, taking into account the inertia of the surrounding air, will lead to the equations of Hydrodynamics, when turbulence is ignored; as intractable in the general case as the unsymmetrical top, although reduced to an elliptic function solution with uniaxial symmetry.

A general discussion of vibration on variational methods tends to become a procession of algebraical equations, leading to no definite result; a concrete case requires to be added to fix the ideas.

Such a graphical picture is obtained for Lagrange's equations of motion of a dynamical system with three degrees of freedom in the statement of the Mathematical Tripos I, May 30, 1907:

"In the uniplanar motion of a body in vibration about a position of stable equilibrium under a conservative system of force:

(i) to make a small statical angular displacement about any point, a small force must be applied to act along the anti-polar of the point with respect to a certain real conic;

(ii) if the conic is projected from a point at a distance k, the radius of gyration of the body, from the plane, the axes of the cone meet the plane in the instantaneous centres of the three principal oscillations."

Application of these principles can be made in Naval Architecture to the change of draft and trim of a ship, and its rolling motion, combination of Statics and Dynamics in Higher Mechanics; this is worked out in a "Note on Ship Geometry" in the *Trans. Inst. Naval Architects*, 1909; here the conic is the momental ellipse of the water line area. G. GREENHILL.

Feb. 9, 1921.

Table des Nombres Premiers et de la Décomposition des Nombres de 1 à 100,000, par G. INGHIRAMI, revue et corrigée et suivie de la Table des Bases des Nombres Tessaréens de 1 à 20,000. Pp. a-l+viii+35. 7.50 fr. 1919. (Gauthier-Villars.)

Table de Caractéristiques de Base 30,030 donnant en un seul coup d'œil les facteurs premiers des nombres premiers avec 30,030 et inférieurs à 901,800,900. By E. LEBON. Tome I. Pp. xxiv+56. 15 fr.+maj. temp. 100%. 1920. (Gauthier-Villars.)

It is to be regretted that more thought was not given to this edition of Inghirami's tables, to which a brief biographical notice is prefixed, for no event less unlikely than the universal desertion of the decimal scale can render the tables themselves obsolete. The particular copy which reached us suffers from defective paging, whereby throughout one sheet the tables that should be face to face are back to back, so that for example the entries 32151-32199 are on the left and face 38101-38149 instead of being on the right and facing 32101-32149. This may be an accident peculiar to a few copies, but the choice of 60 as the number of entries in a column is a fatal error of judgment. If it was out of the question to issue the edition in quarto with the tables sideways, a plan that would have allowed the entries for each group of 10,000 numbers to occupy one double page, it would have been best to be content with 50 entries in a column; as it is, the groups begin at different levels, nor is there even a leaded line or a marginal mark to assist the eye. In the circumstances it has not seemed worth our while to test the accuracy of the tables; the responsible editor, M. Lebon, speaks of mistakes that he has corrected, but does not claim to have checked the figures systematically.

The table of factors is preceded (*not* "followed") by a table of tesseral numbers, to which there is an introduction remarkable for the assertion

that " sans la théorie des nombres, il est impossible de prouver que le rapport de la circonférence au diamètre est incommensurable," and for a savage attack on Gauss.

Nevertheless M. Lebon, in extracting from the *Disquisitiones Arithmeticae* a couple of apt sentences for prelude to his own work, renders unconsciously the homage that he thinks misdirected. In his table of characteristics we have the material for factorising, if not " en un seul coup d'œil," at least with surprisingly little effort, every integer below the square of 30,030 (the product of the primes from 2 to 13), and it is the author's intention to extend the tables to the square of 510,510. The principle, explained with an offensive wealth of eulogistic quotations, is simplicity itself, but the labour of constructing the tables has evidently been heavy, and it is fortunate for mathematicians that M. Lebon's enthusiasm has led him to undertake it.

Good paper, clear type, and ample margins render both books a pleasure to use; had they not been guilty of issuing an imperfect copy—and unless the mail is to blame of ignoring our protest—we should have had nothing but praise for the publishers, who have shown themselves again willing to serve mathematics by the production of unremunerative works. E. H. N.

The Training of Teachers of Mathematics for the Secondary Schools of Countries represented in the International Commission on the Teaching of Mathematics. By R. C. ARCHIBALD. 1918. (Washington, Government Printing Office.)

Belated though this notice of a well-conceived and excellently executed piece of work may be, it comes to us at a time when the question of the teacher is one that is occupying the minds of all concerned with the upbringing of the youth of the country. There is little doubt that during the next few years we shall see great changes in the direction of sending into our secondary schools all who are fit to take advantage of the courses of training they provide. The meshes of the net will be drawn so closely that the potential Faraday will not escape us, and the general level will be slowly and surely raised. A large minority of the taxpayers will object to such " squandering " of the wealth of the community as this implies, and for some time to come the work of administrators will be seriously hampered. The critical question at the moment is the provision of teachers, and the problems of their training will require every light that we can gain from the experience of the past. Hence the value of this carefully compiled little manual which we owe to the untiring labours of one of the most learned and " live " of the Professors across the Atlantic.

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"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."—BACON.

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THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Southampton, Bangor, and Sydney (New South Wales). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

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Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."